LACUNARY STRONGLY CONVERGENT AND LACUNARY STATISTICALLY CONVERGENT DIFFERENCE CLASS OF INTERVAL NUMBERS

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Abstract: In this paper we introduce the notation difference operator Δ_m ($m \geq 0$ be an integer) for studying the properties of interval numbers. Here we study the Lacunary Difference sequence of Interval numbers. We study some algebraic and topological properties of Lacunary Difference sequence of Interval numbers.

Key Words: Lacunary Difference Class, Interval numbers, Solidness, Convergence free.

Subject Classification: 2000 Mathematics 40C05,40J05, 46A45.

1. Introduction

Interval arithmetic was first suggested by Dwyer [5] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [8] in 1959 and Moore and Yang [9] 1962. Furthermore, Moore and others [5], [6], [7] and [10] have developed applications to differential equations. Chiao in [3] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [4] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently Esi in [1] introduced and studied strongly almost−convergence and statistically almost−convergence of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by I$\mathbb{R}$. Any elements of I$\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \mathbb{R}: a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers [5]. Let x_l and x_r be first and last points of $\bar{x}$ interval number, respectively. For x_1, $x_2 \in$ I$\mathbb{R}$, we have $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$,

$$\bar{x}_1 + \bar{x}_2 = \left\{x \in \mathbb{R}: x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\right\} \text{ and if } \alpha \geq 0, \text{ then}$$

$$\alpha\,\bar{x} = \left\{x \in \mathbb{R}: \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\right\}$$

and $\alpha < 0$ then $\alpha\bar{x} = \left\{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\right\}$,

$$\bar{x}_1\bar{x}_2 = \begin{Bmatrix} x \in R: min\left\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\right\} \leq x \leq \\ max\left\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\right\} \end{Bmatrix}$$

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}$$

In the special case $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f : N \to \mathbb{R}$ R by $k \to f(k) = \bar{x}$, $x = (x_k)$. Then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. The $\bar{x}_k$ is called k^{th} term of sequence $\bar{x} = (\bar{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebric properties of w^i can be found in [7].

Now we give the definition of convergence of interval numbers:

A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\bar{x}_k, \bar{x}_o) < \varepsilon$ for all $k \geq ko$ and we denote it by $\lim_k \bar{x}_k = \bar{x}_o$. Thus, $\lim_k \bar{x}_k = \bar{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$

Now we give new definitions for interval sequences as follows:

An interval valued sequence space $\bar{E}$ is said to be solid if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $|\bar{y}_k| \leq |\bar{x}_k|$, or all $k \in N$ and $\bar{x} = (\bar{x}_k) \in \bar{E}$.

An interval valued sequence space $\bar{E}$ is said to be monotone if $\bar{E}$ contains the canonical pre_image of all its step spaces.

An interval valued sequence space $\bar{E}$ is said to be convergence free if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $\bar{x} = (\bar{x}_k) \in \bar{E}$ and $\bar{x}_k = 0$ implies $\bar{y}_k = 0$.

Throughout the paper, $p=(p_k)$ is a sequence of bounded strictly positive numbers. Esi [1] define the following interval valued sequence space:

$$\bar{\ell}(p) = \left\{ \bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} \left[d\left(\bar{x}_{k,} \bar{0}\right) \right]^{p_k} < \infty \right\}$$

and if $p_k = 1$ for all $k \in \mathbb{N}$, then we have

$$\bar{\ell} = \left\{ \bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} \left[d\left(\bar{x}_{k,} \bar{0}\right) \right] < \infty \right\}$$

Kizmaz [2] defined the sequence space for crisp set. This concept further generalized by Tripathy and Esi [12] as follows

Let $m \geq 0$ be an integer then $Z_1(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z_1\}$, for $Z_1 = \ell_\infty$, c and c_0. Where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in \mathbb{N}$ and they showed that these are Banach spaces under the norm $\|x\|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|$. For m = 1, the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are studied by Kizmaz [2].

Tripathy and Baruah [14] defined as follows :

Let $\bar{x}$ = ($\bar{x}_k$) be a sequence of interval numbers and p=(p_k) is a sequence of bounded strictly positive numbers. Let $m \geq 0$ be an integer then $Z(\Delta_m)$ ={($\bar{x}_k$)∈ w^i: $(\Delta_m\bar{x}_k) \in Z$}, for Z = $\bar{\ell}(p)(\Delta_m)$, $\bar{c}(p)(\Delta_m)$ and $\bar{c}_0(p)(\Delta_m)$. Where $\Delta_m\bar{x}_k = \bar{x}_k - \bar{x}_{k+m}$, for all $k\in\mathbb{N}$.

2. Preliminaries

By a lacunary sequence $\theta = (k_r)$, r = 0, 1, 2, 3, 4, . . . where $k_0 = 0$ we mean an increasing sequence of non negative integers with $h_r = k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \dfrac{k_r}{k_{r-1}}$, for r = 1, 2, 3, 4, . . .

Esi [13] defined lacunary strongly convergent interval number and lacunary statiscally convergent interval number as follows

Let $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be any sequence of strictly positive real numbers. A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be lacunary strongly convergent interval number if there is an interval number $\bar{x}_0$ such that

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} \left[d(\bar{x}_k, \bar{x}_0) \right]^{p_k} = 0$$

Here $\bar{x}_k \to \bar{x}_0 \left(\bar{N}_\theta^p\right)$ or $\bar{N}_\theta^p - \lim \bar{x}_k = \bar{x}_0$. Esi defind $\bar{N}_\theta^p$ as lacunary strongly convergent interval numbers. In the special case $\theta = (2^r)$ he defind $\bar{N}^p$ instead of $\bar{N}_\theta^p$.

Let $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be any sequence of strictly positive real numbers. A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be lacunary statically convergent to interval number $\bar{x}_0$ such that

$$\lim_r \frac{1}{h_r} \left| k \in I_r : d(\bar{x}_k, \bar{x}_0) \geq \varepsilon \right| = 0$$

In this Esi write $\bar{x}_k \to \bar{x}_0 \left(\bar{S}_\theta\right)$ then $\bar{S}_\theta - \lim \bar{x}_k = \bar{x}_0$. The set of all lacunary statically convergent to interval number is denoted by $\bar{S}_\theta$. In the special case $\theta = (2^r)$ it is defind by $\bar{S}$.

3. Main Result

In this section we give some definition and proof of this paper.

Let $\theta=(k_r)$ be a lacunary sequence and $p=(p_k)$ be any sequence of strictly positive real numbers. A sequence $\bar{x}=(\bar{x}_k)$ of interval numbers is said to be lacunary Δ_m- strongly convergent interval number if there is an interval number $\bar{x}_0$ such that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[d(\Delta_m \bar{x}_k,\bar{x}_0)\right]^{p_k}=0$$

Here $\bar{x}_k \to \bar{x}_0\left(\bar{N}_\theta^p(\Delta_m)\right)$ or $\bar{N}_\theta^p-\lim \Delta_m \bar{x}_k=\bar{x}_0$. We defind $\bar{N}_\theta^p(\Delta_m)$ as lacunary strongly convergent interval numbers. In the special case $\theta=(2^r)$ he defind $\bar{N}^p(\Delta_m)$ instead of $\bar{N}_\theta^p(\Delta_m)$.

Let $\theta=(k_r)$ be a lacunary sequence and $p=(p_k)$ be any sequence of strictly positive real numbers. A sequence $\bar{x}=(\bar{x}_k)$ of interval numbers is said to be lacunary Δ_m- statically convergent to interval number $\bar{x}_0$ such that

$$\lim_r \frac{1}{h_r}\left|k\in I_r : d(\Delta_m \bar{x}_k,\bar{x}_0)\geq\varepsilon\right|=0$$

In this We write $\bar{x}_k \to \bar{x}_0\left(\bar{S}_\theta \Delta_m\right)$ then $\bar{S}_\theta-\lim \Delta_m \bar{x}_k=\bar{x}_0$. The set of all lacunary statically convergent to interval number is denoted by $\bar{S}_\theta(\Delta_m)$. In the special case $\theta=(2^r)$ it is defind by $\bar{S}(\Delta_m)$.

Theorem 3.1 : The classes $\bar{N}_\theta^p(\Delta_m)$ and $\bar{S}_\theta(\Delta_m)$ are closed under the operations of addition and scalar multiplication

Solution : (i) Let $\bar{x}\in\bar{N}_\theta^p(\Delta_m)$ and $c\in R$

Then $\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[d(c\Delta_m \bar{x}_k, c\bar{x}_0)\right]^{p_k}$

$$= \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[|c| \, d(\Delta_m \bar{x}_k, \bar{x}_0) \right]^{p_k}$$

$$\leq \left\{ \max\left\{1, |c|^H\right\} \right\} \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{x}_k, \bar{x}_0) \right]^{p_k}$$

$$= \left\{ \max\left\{1, |c|^H\right\} \right\} .0$$

$$= 0.$$

(ii) Let $\bar{x}, \bar{y} \in \overline{N}_\theta^p(\Delta_m)$

Then $\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{x}_k \oplus \Delta_m \bar{y}_k, \bar{x}_0 \oplus \bar{y}_0) \right]^{p_k}$

$$\leq K \left[\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{x}_k, \bar{x}_0) \right]^{p_k} + \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{y}_k, \bar{y}_0) \right]^{p_k} \right]$$

$$= 0$$

Therefore

$$\bar{x} \oplus \bar{y} \in \overline{N}_\theta^p(\Delta_m).$$

Theorem 3.2 : The class $\overline{N}_\theta^p(\Delta_m)$ is a Solid in general.

Solution : Let $\bar{x}_k \in \overline{N}_\theta^p(\Delta_m)$ and $\bar{y}_k$ be such that $d\left(\bar{y}_k, \bar{0}\right) \leq d\left(\bar{x}_k, \bar{0}\right)$. Then the result follows from the following inequality

$$\frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{y}_k, \bar{0}) \right]^{p_k} \leq \frac{1}{h_r} \sum_{k \in I_r} \left[d(\Delta_m \bar{x}_k, \bar{0}) \right]^{p_k}$$

Hence $\overline{N}_\theta^p(\Delta_m)$ is a Solid.

Theorem 3.3 : The class $\overline{N}_\theta^p(\Delta_m)$ is not Symmetric in general.

Solution : The result follows from the following example.

Example 1. : Let $m = 2$, $\theta = (2^r)$ and $p_k = 1$ for all $k \in N$.

Consider the sequence $\bar{x}_k = (\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \ldots\ldots)$

Then $\bar{x}_k \in \overline{N}_\theta^p(\Delta_m)$

Then $\bar{y}_k$ be the rearrangement of $\bar{x}_k$ defined by

$$\bar{y}_k = (\bar{1}, \bar{3}, \bar{5}, \bar{6}, \bar{2}, \ldots\ldots)$$

Then clearly $\bar{y}_k \notin \overline{N}_\theta^p(\Delta_m)$

Hence $\overline{N}_\theta^p(\Delta_m)$ is not symmetric in general.

Theorem 3.4 : The class $\overline{N}_\theta^p(\Delta_m)$ is not Convergence free.

Proof . Let $m = 2$, $\theta = (2^r)$ and $p_k = 1$ for all $k \in N$.

We consider the interval sequence $\bar{x} = (\bar{x}_k)$ as follows

$$\bar{x}_k = \left[-\frac{1}{k^2},\ 0\right],$$

Now we have $$\Delta_2 \bar{x}_k = \left[-\frac{1}{k^2},\ \frac{1}{(k+2)^2}\right]$$

Then clearly $\bar{x}_k \in \overline{N}_\theta^p(\Delta_m)$

Again consider the sequence

$$\bar{y}_k = \left[-k^2,\ 0\right]$$

Then

$$\Delta_2 \bar{y}_k = \left[-k^2,\ (k+2)^2\right]$$

Which gives $\bar{y}_k \notin \overline{N}_\theta^p(\Delta_m)$

Hence the class $\overline{N}_{\theta}^{p}(\Delta_m)$ is not Convergence free.

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