A STOCHASTIC MODEL OF WEIGHTED SUJIT DISTRIBUTION WITH APPLICATIONS TO REAL LIFE TIME CANCER DATA

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ABSTRACT

In this study, a new two-parameter lifetime distribution referred to as the Weighted Sujit Distribution (WSD) has been proposed as an extension of the one-parameter Sujit distribution. The proposed model introduces an additional parameter to increase flexibility and improve the modeling of lifetime data. Several important statistical properties of the Weighted Sujit Distribution, including the probability density function (PDF), cumulative distribution function (CDF), reliability analysis, order statistics, moment-generating function, and entropy measures, are derived and discussed. The parameters of the model are estimated using the Maximum Likelihood Estimation (MLE) method. An application to a real-life dataset demonstrates that the proposed Weighted Sujit Distribution provides a superior fit compared to the original Sujit distribution, making it a more flexible and reliable model for analyzing lifetime data in various applied fields such as engineering, medicine, and biological sciences.

KEY WORDS

Weighted Sujit Distribution; Lifetime Data; Reliability Analysis; Hazard Rate Function; Moment Generating Function; Maximum Likelihood Estimation (MLE); Entropy; Order Statistics; Probability Modeling; Two-Parameter Distribution.

INTRODUCTION

Modeling lifetime data plays a vital role in a wide range of applied disciplines such as medical sciences, engineering, reliability analysis, insurance, and finance. In these fields, understanding the behavior of time-to-event data is essential for decision-making, risk assessment, and system optimization. Over the years, several continuous probability distributions have been proposed and extensively used for modeling lifetime data. Among the most commonly used distributions are the exponential, Weibull, gamma, Lindley, and lognormal distributions, along with their various generalizations. Each of these models possesses unique characteristics and is suitable for describing specific types of data behavior.

The exponential distribution, due to its mathematical simplicity, has been widely used in reliability studies. However, its assumption of a constant hazard rate often limits its applicability in real-world situations. The Weibull and gamma distributions offer more flexibility in modeling increasing and decreasing hazard rates, but they sometimes involve computational complexities. The lognormal distribution, although useful in many contexts, lacks a closed-form expression for its survival function, making it less convenient for practical applications.

In this context, the Lindley distribution, introduced by Ghitany et al. (2008), has gained considerable attention due to its simplicity and ability to model monotonic hazard rate functions. It has been widely applied in survival analysis and reliability engineering. Nevertheless, despite its advantages, the Lindley distribution is still limited in capturing more complex hazard rate shapes, such as bathtub or non-monotonic behaviors, which are often observed in real-life data.

To address these limitations, researchers have proposed several new one-parameter lifetime distributions that extend or improve upon the Lindley and exponential models. Notable among these are the Shanker, Aradhana, Devya, Sujatha, Akash, Suja, and Sujit distributions, developed between 2015 and 2017. These distributions provide greater flexibility in modeling lifetime data by incorporating different structural forms and mixture components. In particular, mixture-based distributions have proven to be highly effective in capturing diverse data patterns due to their ability to combine multiple probabilistic behaviors within a single framework.

Among these developments, the Sujit distribution represents an important contribution to the class of one-parameter lifetime models. It is constructed as a mixture of exponential and gamma distributions, which enables it to model a wider range of hazard rate functions, including increasing, decreasing, and bathtub-shaped patterns. Due to this flexibility, the Sujit distribution has demonstrated improved performance compared to traditional models in fitting real-world data. However, like many one-parameter distributions, it may still face limitations when dealing with highly complex or heterogeneous datasets.

One effective approach to enhancing the flexibility of statistical distributions is through the introduction of additional parameters. In particular, weighted distributions provide a powerful framework for modeling data that are subject to bias or unequal probability of observation. The concept of weighted distributions was first introduced by Fisher (1934) and later generalized by Rao (1965). These distributions are especially useful in situations where the probability of observing a unit is proportional to some function of the variable of interest. Applications of weighted distributions are widespread in fields such as reliability theory, survival analysis, biomedical studies, ecology, and branching processes.

Motivated by the need for more flexible and robust lifetime models, the present study proposes a new two-parameter distribution known as the **Weighted Sujit Distribution (WSD)**. This distribution is developed by applying a weighting technique to the existing Sujit distribution, thereby introducing an additional parameter that enhances its adaptability. The inclusion of the weight parameter allows the proposed model to capture a broader spectrum of distributional shapes and hazard rate behaviors, making it more suitable for analyzing complex real-life data.

The statistical properties of the Weighted Sujit Distribution are thoroughly investigated in this study. These include the derivation of the probability density function (PDF), cumulative distribution function (CDF), moments, skewness, kurtosis, and reliability measures such as the hazard rate and mean residual life functions. The flexibility of the proposed model is illustrated through graphical analysis of its density and hazard functions under different parameter values.

Parameter estimation for the proposed distribution is carried out using the Maximum Likelihood Estimation (MLE) method, which is widely recognized for its efficiency and desirable statistical properties. The estimation procedure involves solving non-linear likelihood equations, which can be implemented using numerical techniques such as the Newton–Raphson method in statistical software like R.

To demonstrate the practical applicability of the proposed model, a real-life dataset is analyzed and compared with several existing distributions, including the original Sujit, Lindley, exponential, and other related lifetime models. Model comparison is performed using well-known goodness-of-fit criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), corrected AIC (AICC), and the log-likelihood function. These criteria provide a reliable basis for selecting the best-fitting model, with smaller values indicating better performance.

The results of the data analysis reveal that the proposed Weighted Sujit Distribution consistently provides a better fit compared to the competing models. This improvement is primarily attributed to the additional parameter, which increases the flexibility of the model and allows it to capture the underlying structure of the data more effectively. Therefore, the Weighted Sujit Distribution can be considered a valuable addition to the family of lifetime distributions.

In summary, this study extends the existing Sujit distribution by incorporating a weighting mechanism, resulting in a more flexible and efficient model for lifetime data analysis. The proposed distribution not only retains the desirable properties of the original model but also overcomes its limitations by providing enhanced adaptability. Consequently, it has significant potential for application in various fields, including reliability engineering, medical research, and applied statistics. The Sujit distribution, proposed by Rekha Radhakrishnan et al. (2025), is a new one-parameter continuous distribution developed for modelling lifetime data. It is constructed as a mixture of the Exponential distribution and higher-order Gamma-type components, providing greater flexibility in capturing diverse data patterns without introducing additional parameters. Its Probability Density Function (PDF) is given by

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx \tag{1}$$

The Cumulative distribution function of Sujit distribution (CDF)

$$F(x,\theta) = 1 - \left[1 + \frac{\theta x}{\theta+1}\right]e^{-\theta} \tag{2}$$

The Weighted Sujit Distribution (WSD)

The probability density function of new weighted Sujit Distribution is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}; \lambda > 0, \tag{3}$$

$$f_w(x) = \frac{x^\beta f(x)}{E(x^\beta)}; \lambda > 0$$

Where $w(x)$ be a non-negative weight function and $E(w(x)) = \int w(x)f(x)dx < \infty$.

$$f_w(x) = \frac{x^\beta f(x)}{E(x^\beta)}; \lambda > 0$$

Where,

$$E(x^{\beta}) = \int_0^{\infty} x^{\beta}\, f(x;\theta,\lambda)dx$$

$$= \int_0^{\infty} x^{\beta} \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx$$

$$= \frac{\theta^2}{\theta+1}\int_0^{\infty} x^{\beta}\,(1+x)e^{-\theta x}dx$$

$$= \frac{\theta^2}{\theta+1}\int_0^{\infty} x^{\beta}\, e^{-\theta x}dx + \int_0^{\infty} x^{\beta+1}\, e^{-\theta x}dx$$

$$= \frac{\theta^2}{\theta+1}\int_0^{\infty} x^{(\beta+1)-1}\, e^{-\theta}\ \ dx + \int_0^{\infty} x^{(\beta+2)-1}\, e^{-\theta x}dx$$

$$= \frac{\theta^2}{\theta+1}\left(\frac{\Gamma(\beta+1)}{\theta^{(\beta+1)}} + \frac{\Gamma(\beta+2)}{\theta^{(\beta+2)}}\right)$$

$$= \frac{\theta^2}{\theta+1}\left(\frac{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}{\theta^{(\beta+2)}}\right)$$

$$E(x^{\beta}) = \left(\frac{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}{\theta^{\beta}(\theta+1)}\right) \tag{4}$$

Substitute (1) and (4) in equation (3), we will get the required probability density function of the weighted Sujit distribution as

$$= \frac{x^{\beta}\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx}{\frac{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}{\theta^{\beta}(\theta+1)}}$$

$$f_w(x,\theta,\beta) = \frac{\theta^{\beta+2}}{(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}x^{\beta}(1+x)e^{-\theta x}dx$$

Now, the cumulative distribution function (cdf) of the weighted Sujit distribution is obtained as

$$F_w(x;\theta,\beta) = \int_0^{x} f_w(x;\theta,\beta)dx \tag{5}$$

$$= \int_0^{x} \frac{\theta^{\beta+2}}{(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}x^{\beta}(1+x)e^{-\theta x}dx$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^{x} x^{\beta}(1+x)e^{-\theta x}dx$$

$$\theta x = z \quad \theta dx = dz \quad x = \frac{z}{\theta} \quad dx = \frac{dz}{\theta}$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^{\theta x}\left(\frac{z}{\theta}\right)^{\beta} e^{-z}\frac{dz}{\theta} + \int_0^{\theta x}\left(\frac{z}{\theta}\right)^{\beta+1} e^{-z}\frac{dz}{\theta}$$

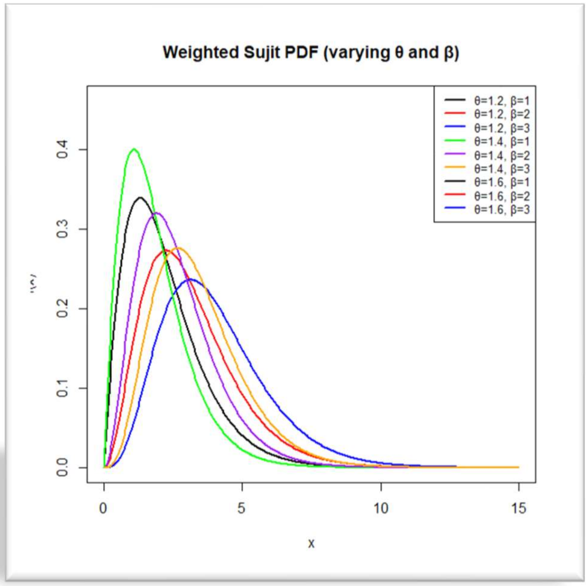
$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^{\theta x}\frac{1}{\theta^{(\beta+1)}}z^{\beta}e^{-z}dz + \int_0^{\theta x}\frac{1}{\theta^{(\beta+2)}}z^{\beta+1}e^{-z}dz$$

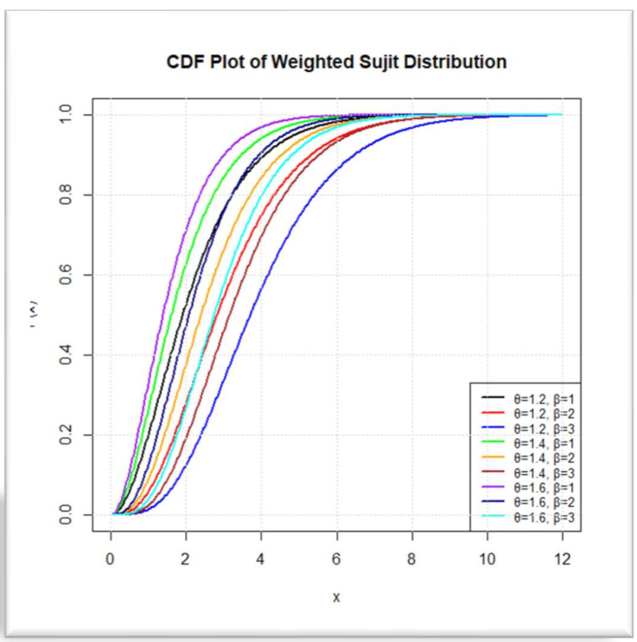
$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\left(\frac{1}{\theta^{(\beta+1)}}\int_0^{\theta x} z^{(\beta+1)-1}e^{-z}dz + \frac{1}{\theta^{(\beta+2)}}\int_0^{\theta x} z^{(\beta+2)-1}e^{-z}dz\right)$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\left(\frac{1}{\theta^{(\beta+1)}}\gamma(\beta+1,\theta x) + \frac{1}{\theta^{(\beta+2)}}\gamma(\beta+2,\theta x)\right)$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\left(\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta^{(\beta+2)}}\right)$$

$$F_w(x;\theta,\beta) = \frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \tag{6}$$





RELIABILITY ANALYSIS

We will discuss the survival function, failure rate, reverse hazard rate and the Mills ratio of the weighted sujit distribution.

Survival Function

The survival function of the weighted Sujit distribution is given by

$$S(x;\theta,\beta) = 1 - F_w(x;\theta,\beta)$$

$$S(x;\theta,\beta) = 1 - \left(\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right) \tag{7}$$

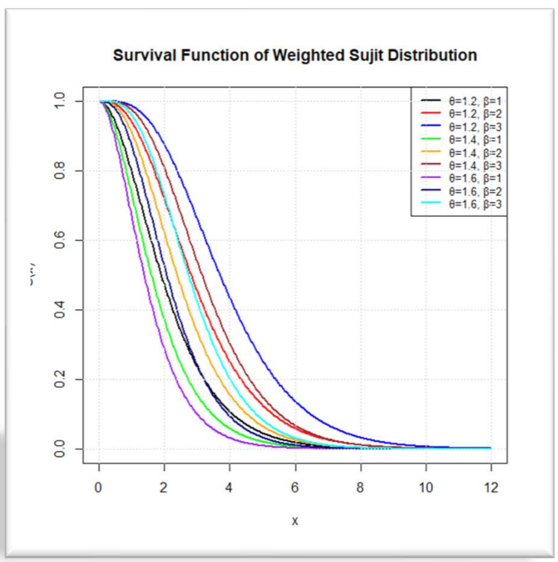
Hazard Function

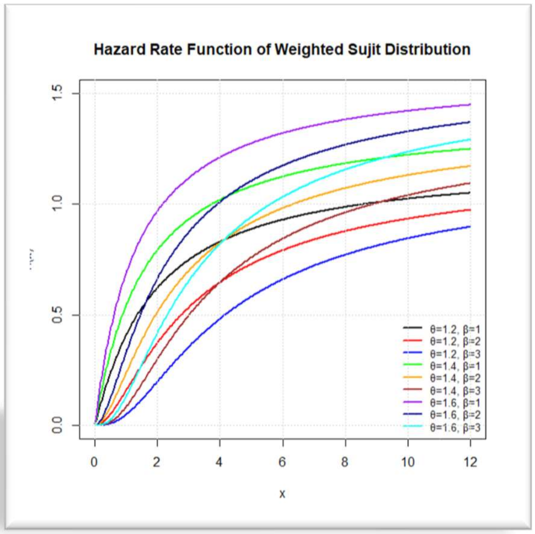
Weighted Sujit distribution in hazard function is given by

$$h(x;\theta,\beta) = \frac{f_w(x;\theta,\beta)}{1-F_w(x;\theta,\beta)}$$

$$h(x;\theta,\beta) = \frac{\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}x^{\beta}(1+x)e^{-\theta}\ dx}{1-\left(\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)}$$

$$h(x;\theta,\beta) = \frac{\theta^{\beta+2}}{(\theta\Gamma(\beta+1)+\Gamma(\beta+2)-\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x))}x^{\beta}(1+x)e^{-\theta x}dx \tag{8}$$





Reverse Hazard Rate Function

$$h_r(x;\theta,\beta)=\frac{f_w(x;\theta,\beta)}{F_w(x;\theta,\beta)}$$

$$h_r(x;\theta,\beta)=\frac{\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}x^{\beta}(1+x)e^{-\theta}\ \ dx}{\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}}$$

$$h_r(x;\theta,\beta)=\frac{\theta^{\beta+2}}{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}x^{\beta}(1+x)e^{-\theta x}dx \qquad (9)$$

Odds Rates Function

$$O(x;\theta,\beta)=\frac{F_w(x;\theta,\beta)}{1-F_w(x;\theta,\beta)}$$

$$O(x;\theta,\beta)=\frac{\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}}{1-\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}}$$

$$o(x;\theta,\beta)=\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)-\theta\ \ (\beta+1,\theta x)+\gamma(\beta+2,\theta x)} \qquad (10)$$

Cumulative hazard Rate Function

$$H(x;\theta,\beta)=-ln(1-F_w(x;\theta,\beta)$$

$$H(x;\theta,\beta)=-ln\left(1-\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right) \qquad (11)$$

Mills Ratio

$$\text{Mills Ratio}=\frac{1}{h_r(x;\theta,\beta)}$$

$$\text{Mills Ratio}=\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta^{(\beta+2)}x^{\beta}(1+x)e^{-\theta}\ \ dx} \qquad (12)$$

MOMENTS AND ASSOCIATION MEASURE

$$E(X^r) = \mu_r' = \int_0^\infty x^r\, f_w(x;\theta,\beta)dx$$

$$= \int_0^\infty x^r \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x^\beta (1+x)e^{-\theta}\ \ dx\right)$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \int_0^x x^r x^\beta (1+x)e^{-\theta x}dx$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \int_0^x x^{r+\beta} e^{-\theta x}dx + \int_0^x x^{r+\beta+1} e^{-\theta x}dx$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \int_0^x x^{(r+\beta+1)-1} e^{-\theta}\ \ dx + \int_0^x x^{(r+\beta+2)-1} e^{-\theta x}dx$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \left(\frac{\Gamma(r+\beta+1)}{\theta^{r+\beta+1}}\right) + \left(\frac{\Gamma(r+\beta+2)}{\theta^{r+\beta+2}}\right)$$

$$= \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \left(\frac{\theta\Gamma(r+\beta+1)+\Gamma(r+\beta+2)}{\theta^{r+\beta+2}}\right)$$

$$\mu_r' = \frac{\theta\Gamma r+\beta+1+\Gamma r+\beta+2}{\theta^r(\theta\Gamma(\beta+1)+\Gamma(\beta+2))} \tag{13}$$

Put $r = 1$ and $r = 2$ we get the moments and variance of newly weighted Sujit distribution is,

$$\mu_1' = \frac{\theta\beta+2\beta}{\theta(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}$$

$$\mu_2' = \frac{2\theta\beta+6}{\theta^2(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}$$

Variance

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \left(\frac{2\theta\beta+6\beta}{\theta^2(\theta\Gamma(\beta+1)+\Gamma(\beta+2))} - \left(\frac{\theta\beta+2\beta}{\theta(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}\right)^2\right) \tag{14}$$

Standard Deviation

Standard deviation $= \sqrt{\text{variance}}$

$$\sigma = \sqrt{\left(\frac{2\theta\beta+6\beta}{\theta^2(\theta\Gamma(\beta+1)+\Gamma(\beta+2))} - \left(\frac{\theta\beta+2\beta}{\theta(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}\right)\right)}$$

Moment Generating Function

$M_x(t) = E(e^{tx})$

$M_x(t) = \int_0^\infty e^{tx} f_w(x;\theta,\beta)dx$

Using Taylor's series expansion

$M_X(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right]$

$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right) \int_0^\infty f_w(x;\theta,\beta)dx$

$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right) \mu_r{}'$

$$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right)\left(\frac{\theta\Gamma(r+\beta+1)+\Gamma(r+\beta+2)}{\theta^r(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}\right) \tag{15}$$

Similarly, the Characteristics function of the new weighted Sujit Distribution can be obtained by

$\phi_x(it) = E(e^{itx})$

$\phi_x(it) = \int_0^\infty e^{itx} f_w(x;\theta,\beta)dx$

$\phi_x(it) = \sum_{r=0}^\infty \left(\frac{it^r}{r!}\right)\left(\frac{\theta\Gamma(r+\beta+1)+\Gamma(r+\beta+2)}{\theta^r(\theta\Gamma(\beta+1)+\Gamma(\beta+2))}\right)$

Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be the order statistics of the continuous population with probability density function $f_w(x)$ and cumulative distribution function with $F_w(x)$. Then the probability density function of r^{th} order statistics $X_{(r)}$ is given by,

$$f_{w(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_w(x)[F_w(x)]^{r-1}[1-F_w(x)]^{n-r}$$

The probability density function of r^{th} order statistics $X_{(r)}$ of the weighted sujit distribution is given by

$$f_{w(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x^\beta(1+x)e^{-\theta x}dx\right) \times \left(\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{r-1} \times \left(1 - \frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{n-r} \tag{16}$$

Therefore, the Probability density function of first Order Statistics X_n of weighted Sujit distribution is can be obtained as

$$f_{w(1)}(x) = \frac{n(n-1)!}{(1-1)!(n-1)!}\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x^\beta(1+x)e^{-\theta x}dx\right) \times \left(1 - \frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{n-r}$$

$$f_{w(n)}(x) = \frac{n!}{(n-1)!(1-1)!}\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x^\beta(1+x)e^{-\theta}\ dx\right) \times \left(\frac{\theta\gamma(\beta+1,\theta x)+\gamma(\beta+2,\theta x)}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{n-1}$$

Likelihood Ratio Test

Let $X_1, X2_{,...,}X_n$ be a random sample from the new Sujit distribution. To test the hypothesis

$H_0: f(x) = f(x;\theta)$ against $H_1: f_w(x) = f_w(x;\theta,\beta)$

In order to test whether the random sample of size n comes from the weighted Sujit distribution, the following test statistics is used by

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_w(x;\theta,\beta)}{f(x;\theta)}$$

$$= \prod_{i=0}^{n} \left(\frac{\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x^{\beta}(1+x)e^{-\theta x} dx}{\frac{\theta^2}{\theta+1}(1+x)e^{-\theta}\ dx} \right)$$

$$= \prod_{i=0}^{n} \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right) x_i^{\beta}$$

$$= \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right)^n \prod_{i=0}^{n} x_i^{\beta}$$

We should reject the null hypothesis, if

$$\Delta = \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right)^n \prod_{i=0}^{n} x_i^{\beta} > k \text{ (or)}$$

Equivalently, We shall reject the null hypothesis, if

$$\Delta^* = \prod_{i=0}^{n} x_i^{\beta} > k \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right)^n$$

$$\Delta^* = \prod_{i=0}^{n} x_i^{\beta} > k^* \ where,$$

$$k^* = k \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right)^n$$

Then

$p(\Delta^* > \lambda^*)$, where, $\lambda^* = \prod_{i=0}^{n} x_i^{\beta}$ is less than a specified level of significance, and $\prod_{i=0}^{n} x_i^{\beta}$ is the observed value of Δ^*

MAXIMUM LIKELIHOOD ESTIMATE AND FISHER INFORMATION MEASURE

$$L(x) = \prod_{i=1}^{n} f_w\,(x)$$

$$L(x) = \prod_{i=1}^{n} \frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} x_i^{\beta}(1+x)e^{-\theta xi}$$

$$L(x) = \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \right)^n \prod_{i=1}^{n} x_i^{\beta}(1+x)e^{-\theta xi} \tag{17}$$

The log likelihood function is given by

$$= n(\beta+2)log(\theta) - n\log\left(\theta^{\beta+1} + \theta^{\beta+2}\right) + \beta \sum_{i=1}^{n} log\, x_i + \sum_{i=1}^{n} log\,(1+x_i) - \theta \sum_{i=1}^{n} x_i$$

Now, differentiating the log likelihood equation (17)

$$\frac{\partial logL}{\partial\theta}=\frac{n(\beta+2)}{\theta}-n\left(\frac{(\beta+1)\theta^{\beta}+(\beta+2)\theta^{\beta+1}}{\theta^{\beta+1}\theta^{\beta+2}}\right)-\sum_{i=1}^{n}x_i$$

$$\frac{\partial logL}{\partial\beta}=nlog(\theta)-n\left(\frac{\theta^{\beta+1}log\theta+\theta^{\beta+2}log\theta}{\theta^{\beta+1}\theta^{\beta+2}}\right)+\sum_{i=1}^{n}\log x_i$$

BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves are used in economics in order to study income, etc., but they are used in other fields like demography, insurance, medicine, and reliability. The Bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^q xf(x)dx$$

$$B(p)=\frac{1}{p\mu_1'}\int_0^q xf(x;\theta,\beta)dx \text{ and}$$

$$L(p)=\frac{1}{\mu_1'}\int_0^q xf(x;\theta,\beta)dx$$

Where, $q=F^{-1}(p);\ q\epsilon[0,1]$ and $\mu=(x)$

Hence, the Bonferroni and Lorenz curves of our distribution are given by,

$$B(p)=\int_0^q x\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}x^{\beta}(1+x)e^{-\theta x}dx$$

$$=\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^q x^{\beta+1}(1+x)e^{-\theta x}dx$$

$$=\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^q x^{\beta+1}\ e^{-\theta x}dx+\int_0^q x^{\beta+2}e^{-\theta}\ dx$$

Put, $\theta x=t,\qquad x=\frac{t}{\theta},\qquad dx=\frac{dt}{\theta}$

when $x\to 0, t\to 0$, and $x\to x, t\to\theta x$

$$=\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\int_0^q\left(\frac{t}{\theta}\right)^{\beta+1}e^{-t}\frac{dt}{\theta}+\int_0^q\left(\frac{t}{\theta}\right)^{\beta+2}e^{-t}\frac{dt}{\theta}$$

After the simplification, we get

$$B(p)=\frac{\left(\theta^{\beta+2}\gamma(\beta+2,\theta q)+\theta^{\beta+2}\gamma(\beta+3,\theta q)\right)}{\theta^{\beta}\theta\Gamma(\beta+1)+\Gamma(\beta+2)}$$

Where,

$$L(p)=B(p)$$

$$L(p)=\frac{\left(\theta^{\beta+2}\gamma(\beta+2,\theta q)+\theta^{\beta+2}\gamma(\beta+3,\theta q)\right)}{\theta^{\beta}\theta\Gamma(\beta+1)+\Gamma(\beta+2)} \tag{18}$$

ENTROPIES

Entropy is important in different areas such as probability and economics, communication theory, physics, and statistics. Entropies are applied to quantify a system's diversity, uncertainty, or randomness. An indicator of the uncertainty's variation is the entropy of a random variable X.

Renyi Entropy

The Renyi entropy is significant as a diversity index. The Renyi entropy is also important in quantum information. It can be used as a measure of entanglement for a given probability distribution. Renyi entropy is given by

$$R_\beta = \frac{1}{1-\lambda} log \int_0^\infty \left(f_w(x)\right)^\lambda dx \; ; \; \beta > 0, \beta \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(f_w(x;\theta,\beta)\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right) x^{\lambda\beta}(1+x)e^{-\theta x}dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^\lambda \int_0^\infty x^{\lambda\beta}(1+x)^\lambda\, e^{-\lambda} \quad dx \tag{19}$$

Using binomial expansion

$$(1+x)^\lambda = \sum \binom{n}{r} a^{n-r} d^r$$

$$R_\lambda = \sum_{i=0}^{\lambda} \sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} \int_0^\infty x^{\lambda\beta} x^{(i-j)}$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^\lambda \sum_{i=1}^{\lambda} \sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} \int_0^\infty x^{(\lambda+\beta+i-j)} e^{-\lambda} \quad dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^\lambda \sum_{i=1}^{\lambda} \sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} \frac{\Gamma\lambda+\beta+i-j+1}{(\theta\beta)^{\lambda+\beta+i-j+1}}$$

Tsallis Entropy.

The Boltzmann-Gibbs (B-G) statistical property generalization initiated by Tsallis has received a great deal of attention. This B-G statistic was first introduced as the mathematical expansion of Tsallis entropy (Tsallis, 1988) for continuous random variables; this generalization of B-G was introduced in order to suggest. Which is defined as

$$T_\lambda = \frac{1}{1-\lambda}\left(1 - \int_0^\infty \left(f_w(x;\theta,\beta)\right)^\lambda dx\right) \lambda > 0, \lambda \neq 1$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1 - \int_0^\infty \left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right) x^{\lambda\beta}(1+x)\, e^{-\theta} \;\right)^\lambda dx$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1-\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{\lambda}\int_0^\infty x^{\lambda\beta}(1+x)^{\lambda}\, e^{-\lambda\theta x}dx\right) \tag{20}$$

Using binomial expansion

$$T_\lambda = \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\int_0^\infty x^{\lambda\beta}x^{(i-j)}$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1-\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{\lambda}\sum_{i=1}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\int_0^\infty x^{(\lambda+\beta+i-j)}e^{-\lambda\theta x}\, dx\right)$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1-\left(\frac{\theta^{\beta+2}}{\theta\Gamma(\beta+1)+\Gamma(\beta+2)}\right)^{\lambda}\sum_{i=1}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\frac{\Gamma\lambda+\beta+i-j+1}{(\theta\beta)^{\lambda+\beta+i-j+1}}\right)$$

DATA ANALYSIS

1.Consider a simulated data represents the survival time (in days) of 73 patients who diagnosed with acute bone cancer, as follows,

0.09	**0.76**	**1.81**	**1.10**	**3.72**	**0.72**	**2.49**	**1.00**	**0.53**	**0.66**
31.61	**0.60**	**0.20**	**1.61**	**1.88**	**0.70**	**1.36**	**0.43**	**3.16**	**1.57**
4.93	**11.07**	**1.63**	**1.39**	**4.54**	**3.12**	**76.01**	**1.92**	**0.92**	**4.04**
1.16	**2.26**	**0.20**	**0.94**	**1.82**	**3.99**	**1.46**	**2.75**	**1.38**	**2.76**
1.86	**2.68**	**1.76**	**0.67**	**1.29**	**1.56**	**2.83**	**0.71**	**1.48**	**2.41**
0.66	**0.65**	**2.36**	**1.29**	**13.75**	**0.67**	**3.70**	**0.76**	**3.63**	**0.68**
2.65	**0.95**	**2.30**	**2.57**	**0.61**	**3.93**	**1.56**	**1.29**	**9.94**	**1.67**
1.42	**4.18**	**1.37**							

2. This data consists of the life time (in years) of 39 cancer (leukemia) patients from one of ministry of health hospitals in soudhi arabia reported in (25). This actual data is,

0.315	0.496	0.616	1.145	1.208	1.263	1.414
2.025	2.036	2.162	2.211	2.370	2.532	2.693
2.805	2.910	2.912	3.192	3.263	3.348	3.427
3.499	3.534	3.767	3.751	3.858	3.986	4.049
4.244	4.323	4.381	4.392	4.397	4.647	4.753
4.929	4.973	5.074	5.381			

We are using the criteria values, like AIC (Akaike information criterion), $AICC$ (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser values of $AIC, AICC, BIC$ and $-2\log L$ can be evaluated by using the formulas as follows:

$$AIC = 2K - 2\log L \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \text{ and } BIC = k\log n - 2\log L$$

Where, $K =$ number of parameters, n samplesize and $-2\log L$ is the maximized value of loglikelihood function.

Table1; $MLEs\ AIC, BIC, AICC,$ and $-2\log L$ of the fitted distributions for given data set

Distribution	ML Estimates	-2logL	AIC	BIC	AICC
Weighted Sujit	$\hat{\theta}=0.26222684(0.04363028)$ $\hat{\beta}=0.43819675(0.32241403)$	389.7811	393.7811	398.0989	393.9525 2
Sujit	$\hat{\theta}=0.20971909(0.01867716)$	392.0752	394.0752	396.234	394.1315 3
Remkan	$\hat{\eta}=0.83422230(0.05777621)$ $\hat{\phi}=10.07023399(7.00908233)$	461.7422	465.7422	470.3232	465.9136 2
Rama	$\hat{\theta}=0.4539552(0.0280582)$	406.6495	408.6495	410.8084	408.7058 3
Fuyi	$\hat{\theta}=1.00100000(0.04475047)$	528.4208	530.4208	532.5796	530.4771 3
Exponencial	$\hat{\theta}=0.11481347(0.01435059)$	405.0524	407.0524	409.2113	407.1087 3
Pranav	$\hat{\theta}=0.45759584(0.02828821)$	407.6214	409.6214	411.7802	409.6777 3
Lindley	$\hat{\theta}=0.20971909(0.01867716)$	392.0752	394.0752	396.234	394.1315 3
Akash	$\hat{\theta}=0.71628684(0.04656348)$	419.7666	421.7666	424.0571	421.8229 3

Table2; $MLEs\ AIC, BIC, AICC,$ and $-2\log L$ of the fitted distributions for given data set

Distribution	ML Estimates	-2logL	AIC	BIC	AICC
Weighted Sujit	$\hat{\theta}=1.1595244(0.2509973)$ $\hat{\beta}=1.9198255(0.7193915)$	143.1436	147.1436	150.4707	147.4769
Sujit	$\hat{\theta}=0.5277071(0.0616171)$	156.5028	158.5028	160.1664	158.6109
Akash	$\hat{\theta}=0.80168365(0.07120997)$	149.0561	151.0561	152.7196	151.1642
Lindley	$\hat{\theta}=0.5277071(0.0616171)$	156.5028	158.5028	160.1664	158.6109
Exponential	$\hat{\theta}=0.31893857(0.05107054)$	167.1353	169.1353	170.7988	169.2434
Two parameter Pranav	$\hat{\theta}=1.1595244(0.2509973)$ $\hat{\beta}=1.9198255(0.7193915)$	461.7422	147.1436	150.4707	147.4769
Power Akash	$\hat{\alpha}=0.89626507(0.08584829)$ $\hat{\theta}=0.88460848(0.10560663)$	147.6645	151.6645	154.9916	151.9978
Weighted Power Akash	$\hat{\alpha}=20.8171738(65.0867502)$ $\hat{\theta}=2.7879442(0.5815882)$ $\hat{c}=15.9940081(4.8835257)$	156.5028	229.332	235.5647	230.0177 1

Interpretation of Results

From Table 1 and Table 2, the performance of different distributions is compared using model selection criteria such as −2logL, AIC, BIC, and AICC, where smaller values indicate a better fit.

In Table 1, the Weighted Sujit Distribution (WSD) shows the lowest −2logL (389.7811), AIC (393.7811), BIC (398.0989), and AICC (393.95252) compared to all other competing models such as Sujit, Lindley, Exponential, and Akash distributions. This clearly indicates that the Weighted Sujit model provides the best fit for the first dataset. Although the Sujit and Lindley distributions perform reasonably well, their AIC and BIC values are slightly higher, confirming that incorporating the weight parameter improves model flexibility and fitting accuracy.

Similarly, in Table 2, the Weighted Sujit Distribution again produces the smallest values of −2logL (143.1436), AIC (147.1436), BIC (150.4707) and AICC (147.4769) among all considered distributions. This further strengthens the evidence that the proposed model consistently outperforms the existing models. Other models such as Akash and Power Akash show competitive results but still have higher information criteria values than WSD.

Overall, the results from both datasets demonstrate that the Weighted Sujit Distribution provides a better representation of the data, due to its additional parameter which increases flexibility in capturing variability and distributional shape. Hence, it is more suitable for modelling real-life lifetime data compared to traditional one-parameter distributions.

CONCLUTION

Based on the comparative analysis presented in Tables 1 and 2, it is evident that the proposed Weighted Sujit Distribution (WSD) provides the best fit among all the considered models. This is supported by its consistently lower values of −2logL, AIC, BIC, and AICC, which indicate superior model performance.

The inclusion of an additional parameter in the Weighted Sujit Distribution enhances its flexibility, allowing it to capture the underlying characteristics of the data more effectively than the existing one-parameter and other competing distributions. While models such as Sujit, Lindley, and Akash distributions show reasonably good fits, they are outperformed by the proposed model.

Therefore, the Weighted Sujit Distribution can be regarded as a more efficient and reliable model for analyzing lifetime data. Its improved fitting ability makes it highly suitable for practical applications in areas such as reliability analysis, survival studies, and applied statistical modeling.

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