NEW EXACT ROTATIONAL AS WELL AS TRANSLATIONAL INVARIANT SOLUTIONS OF GENERALIZED LIOUVILLE'S EQUATION

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Abstract:-

Exact rotational and translational invariant solutions of the generalized Liouville's equation $u_{,tt} - u_{,xx} = \alpha e^{ku}$ found by using Lie's infinitesimal continuous similarity transformation method. Solutions are valid for all values of the parameter k except k$= 0$, for k$= 1$ these solutions are already reported. Solutions are different from known solutions obtained by Backlund transformation. Translational invariant exists only along with rotational invariant part whereas, the pure rotational invariant solutions are possible without translational invariant part. As such solutions these may be suitable for modeling mean field vortex in steady flow.

Key words:

Similarity transformation, Hyperbolic, Gaussian curvature, Eucledian metric, Backlund transformation, Lax pairs, Integrability, Rotaional, Translational, .Infinitesimal, Symmetry, Vortex.

1. **Introduction**

Liouville's equation is a nonlinear partial differential equations with two independent variables introduced by Liouville [1] that describes a constant Gaussian curvature conformal to the restrictions of the Eucleadian metric of two dimensional surface. In mathematical physics [2]- [8] this equation appears in the context of the mean field vortex in the Electroweak theory [09],[10],[11]. Backlund transformation of this equation is reported [08]'[09]. Existence of Lax pair and inverse scattering transformation also confirmed, as such Liouville's equation is considered as a completely integrable system. This equation is a special case of Toda lattice equation.

The well known form of the Lioville's equation with two independent variables x and t is,

$$u_{tt} - u_{xx} = \alpha e^{u} . \qquad (1)$$

where α *is* $a$parameter. Equation (1) is a hyperbolic nonlinear partial differential equation. The generalized Liouville's equation with two parameters is,

$$u_{tt} - u_{xx} = \alpha e^{kx} , \qquad (2)$$

where α and k are nonzero parameters. For k$= 1$ this is same as equation (1).Using Lie's infinitesimal continuous parameter similarity transformation (LST) the exact solutions with rotational as well as translational solutions are already reported [15]. This study is reporting new exact rotational as well as translational invariant solutions of generalized Liouville's equation (2) using LST method .

02. Lie's Similarity Transformation Method of Solving Partial differential Equations

Brief outline of the LST method of solving PDE by reducing the number of independent variables by one is the following [14]. Let the given PDE with n number of independent variables and m dependent variables be,

$$\psi(x_1,x_2,\ldots.x_n,\omega^1,\omega^2,\ldots.. \omega^m, \omega^i_{,k}) = 0 , \tag{3}$$

Where x_j are independent variables and ω^i are dependent variables, i=1,2,…n and j=1,2,…m.and $\omega^i_{,k}$ is the partial differential of dependent variable ω^j with respect independent variable x_k. Apply, one infinitesimal parameter ϵ>0, family of infinitesimal transformations,

$$x_j=x_j + \epsilon X_j(x_i,\omega^i)+ O(\epsilon^2), \tag{4}$$

$$\omega^i=\omega^i + \epsilon W^i(x_j,\omega^i) + O(\epsilon^2) , \tag{5}$$

Correspondingly ,the first order partial derivatives of dependent variables ω^i are transformed as,

$$\omega^i_{,k} = \omega^i_{,k} + \epsilon[W^i_{,k}] + O(\epsilon^2), \tag{6}$$

Where $[W^i_{,k}]$ is called 'first extension' of first order derivatives $\omega^i_{,k}$ of dependent variables ω^i and its expansion as [14],

$$[W,^i_{,k}] = \frac{\partial\omega^i}{\partial x_k} + \frac{\partial W^i}{\partial\omega^\mu}\omega^\mu_{,k} - \frac{\partial X_\nu}{\partial x_k}\omega^i_{,\nu} - \frac{\partial X_\nu}{\partial\omega^\mu}\omega^\mu_{,k}\omega^i_{,\nu} \quad . \tag{7}$$

The second partial derivatives $\omega^i_{,jk}$ with respect to the variables x_j and x_k are transformed as,

$$\omega^i_{,jk} = \omega^i_{,jk} + \epsilon[W,^i_{jk}] + O(\epsilon^2), \tag{8}$$

where $[W^i_{,jk}]$ is called 'second extension' of the second order derivatives $\omega^i_{,jk}$ of the dependent variables ω^i with respect to the independent variables x_j and x_k, and its expansion is,

$$[W^i_{,jk}] = \frac{\partial^2 W^i}{\partial x_j \partial x_k} + \frac{\partial^2 W^i}{\partial x_j \partial w^\mu}\omega^\mu_{,k} + \frac{\partial^2 W^i}{\partial x_k \partial \omega^\mu}\omega^\mu_{,j} - \frac{\partial^2 X_\nu}{\partial x_j \partial x_k}\omega^i_\nu + \frac{\partial W^i}{\partial \omega^\mu}\omega^\mu_{,jk}$$

$$- \frac{\partial X_\nu}{\partial x_j}\omega^i_{,k\nu} - \frac{\partial X_\nu}{\partial x_k}\omega^i_{,j\nu} + \frac{\partial^2 W^i}{\partial \omega^\lambda \partial \omega^\mu}\omega^\lambda_{,j}\,\omega^\mu_{,k} - \frac{\partial^2 X_\nu}{\partial x_j\,\partial \omega^\mu}\omega^\mu_{,j}\omega^i_{,\nu} - \frac{\partial^2 X_\nu}{\partial x_k \partial \omega^\mu}\omega^\mu_{,k}\omega^i_{,\nu}$$

$$\frac{\partial X_\nu}{\partial \omega^\mu}[\omega^i_{,\nu}\omega^\mu_{,jk} + \omega^\mu_{,j}\omega^i_{,\nu k} + \omega^\mu_{,k}\omega^i_{,j\nu}] - \frac{\partial^2 X_\nu}{\partial \omega^\lambda \partial \omega^\mu}\omega^\lambda_{,k}\omega^\mu_{,j}\omega^i_{,\nu}\,. \tag{9}$$

The so called " invariant surface condition"[14] for a similarity transformation , (4) and (5) of the given PDE (3) for a second order PDE is

$$X_i \frac{\partial \psi}{\partial x_i} + W^i \frac{\partial \psi}{\partial \omega^i} + [W^i_{,j}]\frac{\partial \psi}{\partial \omega^i_{,j}} + [\mathrm{W}^{\mathrm{i}}_{,\mathrm{jk}}]\frac{\partial \psi}{\partial \omega^i_{,jk}} = 0. \tag{10}$$

On solving above" invariant condition" the similarity transformation under which the given PDE (3) can be transform into another PDE with one independent variables less can be found out.

Then for finding the similarity transformation solve,

$$X_j \frac{d\omega^i}{dx_j} + W^i \frac{d\omega^i}{dW^i} = 0\ , \tag{11}$$

using the Lagrange's conditions,

The solution of (10) gives the similarity variables η_j as function of independent variables x_j and arbitrary constants ,and the 'similarity solutions' ω^i as function of similarity variables η_j. On substituting similarity solution in the given PDE (3) the resultant equation is called similarity reduced equation having one independent variable less than that of given PDE . All exact solutions of similarity reduced equation is also exact solutions of given PDE (3) ,but converse need not be satisfied.

3. Lie's Similarity Transformation of generalized Liouville's equation

Corresponding to the (10) for the generalized Liouville's equation (2) gives

$$[U,_{tt}] - [U,_{xx}] = \alpha k\,[U] e^{ku} \quad . \qquad (11)$$

Substitute the expansions of second 'extensions' (9) for $[U_{,tt}]\ annd\ [U_{,xx}]$ in equation (11) and equate coefficients of different orders of derivatives of dependent variables u yields following essential constrained equations,

$$U,_{tt} -, = \alpha kU\, e^{ku}, \quad , -2U,_{xu} - X,_{tt} = 0, \quad 2U,_{tu} - T_{,tt} + T_{,xt} = 0\ ,$$

$$T,_{x} - X,_{t} = 0, \quad T_{,xu} - X,_{tu} = 0, \quad T,_{t} = T,_{u} = X,_{x} = X,_{u} = U = 0. \qquad (12)$$

On solving the above constrained equations (12) yield ,

$$X = ct + \omega\ , \quad T = -cx + \kappa, \quad U = 0\ , \qquad (13)$$

where κ , $c\ and\ \omega$ are arbitrary constants of integration.

The similarity variable with respect to the (13) can be found out by (11) as,

$$\frac{dx}{X} = \frac{dt}{T} = \frac{dU}{U}\ . \qquad (14)$$

On substitute X, T, U from (13) gives the similarity variable h(x,t) as

$$h(x, t) = [\, c(x^2 - t^2) + (\kappa\, x - \omega t) \; + \frac{(\kappa^2 - \omega^2)}{2c}] \; . \qquad (15).$$

Similarity solution is corresponding to the generalized Liouville's equation (2) is

$$u(x\, , t) = h(x\, , t). \qquad (16)$$

Substitute above similarity solution (16) in the (2) yields the similarity reduced equation corresponds to the generalized Liouvillie's equation (2) as

$$h\frac{d^2u}{dh^2} + \frac{du}{dh} = -\frac{\alpha}{2c}e^{ku}\, . \qquad (17)$$

Possible exact solutions of above similarity reduced equation (17) is obtained as

$$\exp(u) = (\frac{2Bh^{(n-1)}}{[B+Ah^n]^2})^{\frac{1}{k}} \; , \qquad (18)$$

where A , B and n are arbitrary constants of integration such that A$\neq 0\; B \neq 0$

and n$\neq 0 \; and \; n \neq 1$.

4. Discussion

Liouville's equation with two independent variables is a well studied system. Clarin [16],[17] reported a type of general solution of this equation using Backlund transformation by connecting solutions of d'Alembert's euation and Liouville's euation but solutions have singularities. Whereas, the solutions of this study (18) are nonsingular when the arbitrary constant B is positive and nonzero **values and the**

values of n are even. For k=1 bove solution (18) is already reported [15].

When $\omega = \kappa = 0$ gives pure rotational invariant solution with out translational invariant part.Whereas, c $\neq 0$ hence pure translational invariant part can not be obtained. So this new solutions represents stationary as well as moving with translational invariant vortex in a fluid. Since Liouville's equation having both Backlund transformation and Inverse Scattering transformations above solutions are of the class of solitons . All known solutions of solitons are functions of exponential functions but above new solutions are independent of exponential functions.

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