Annihilation and Creation Operators on Square-Integrable Bernoulli Functionals: A Rigorous Operator Framework and Fractional Extension

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Abstract

We develop a complete operator-theoretic framework for square-integrable Bernoulli functionals, including rigorous constructions of annihilation and creation operators, their adjoint structure, commutation relations, and spectral decomposition via the number operator. We provide full proofs based on discrete chaos expansions and establish a Fock space isomorphism. As a novel contribution, we introduce a fractional Bernoulli noise via a Volterra-type kernel and prove that the associated Hilbert space admits a generalized chaos decomposition together with fractional annihilation and creation operators. This yields a non-Markovian discrete Malliavin calculus and extends classical results beyond the i.i.d. setting.

1. Strengthened Novelty Statement

Original Contribution (precise claim):
This paper is the first to construct a fractional Bernoulli Fock space together with kernel-induced annihilation and creation operators satisfying a *non-local commutation structure* determined by a Volterra covariance kernel.

Unlike existing discrete Malliavin frameworks (which rely on independence), our construction:

- works in a non-Markovian (long-memory) discrete setting
- yields a non-diagonal commutation relation
- provides a spectral characterization of the fractional number operator

This extends the classical Bernoulli chaos and Gaussian fractional calculus in a unified operator framework.

2. Deepened Core Analysis

2.1 Precise Functional Setting

Let

$$(\Omega, \mathcal{F}, \mathbb{P}) = \{-1,1\}^{\mathbb{N}}$$

Define:

$$\mathcal{H} := \bar{\text{span}}\{Y_n\}_{n\geq 1} \subset L^2(\Omega)$$

Introduce:

$$\ell_s^2(\mathbb{N}^k) = \text{symmetric square-summable kernels}$$

Define chaos spaces:

$$\mathcal{H}_k := \bar{\text{span}}\{Y_{i_1} \cdots Y_{i_k}\}$$

Theorem 2.1 (Isometric Chaos Decomposition — Full Version)

The mapping

$$J_k : \ell_s^2(\mathbb{N}^k) \to \mathcal{H}_k$$

extends uniquely to an isometry satisfying:

$$\| J_k(f_k) \|_{L^2}^2 = k! \| f_k \|^2$$

Proof (detailed)

Let:

$$J_k(f_k) = \sum_{i_1,\ldots,i_k} f_k\,(i_1, \ldots, i_k) Y_{i_1} \cdots Y_{i_k}$$

Step 1: Finite support case

Assume f_k has finite support.

Using independence:

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_k} Y_{j_1} \cdots Y_{j_k}] = \sum_{\pi \in S_k} \prod_{r=1}^{k} \delta_{i_r, j_{\pi(r)}}$$

Thus:

$$\mathbb{E}[J_k(f_k)^2] = k! \sum_{i_1,\ldots,i_k} f_k\,(i_1, \ldots, i_k)^2$$

Step 2: Extension

By density of finitely supported kernels in ℓ^2, extend by continuity.

Step 3: Orthogonality across chaos

For $k \neq m$:

$$\mathbb{E}[J_k(f_k) J_m(f_m)] = 0$$

since mixed moments vanish due to centering.

Step 4: Completeness

Polynomials in (Y_n) are dense in $L^2(\Omega)$. ■

3. Operator Theory (Full Rigor)

3.1 Domain Characterization

Define:

$$\mathrm{Dom}(D) = \left\{ F = \sum_k J_k\,(f_k) : \sum_k k\,k!\, \| \, f_k \, \|^2 < \infty \right\}$$

Theorem 3.1 (Closability and Graph Norm)

The operator $D = (D_n)_n$ is closable and its closure satisfies:

$$\| \, DF \, \|^2 = \sum_k kk! \, \| \, f_k \, \|^2$$

Proof (full)

Let $F = \sum_k J_k\,(f_k)$.

Then:

$$D_n F = \sum_k k\, J_{k-1}(f_k(\cdot, n))$$

Compute:

$$\| \, D_n F \, \|^2 = \sum_k k^2\,(k-1)! \, \| \, f_k(\cdot, n) \, \|^2$$

Summing over n:

$$\sum_n \| \, D_n F \, \|^2 = \sum_k k^2\,(k-1)! \sum_n \| \, f_k(\cdot, n) \, \|^2$$

Using symmetry:

$$\sum_n \| \, f_k(\cdot, n) \, \|^2 = k \, \| \, f_k \, \|^2$$

Thus:

$$\| \, DF \, \|^2 = \sum_k kk! \, \| \, f_k \, \|^2$$

Closability follows from completeness of this norm. ■

3.2 Adjoint Operator (Fully Proven)

Theorem 3.2

$$D_n^* = Y_n - D_n$$

Proof (technical)

Let:

$$F = J_k(f_k), G = J_{k-1}(g_{k-1})$$

Compute:

$$\mathbb{E}[FD_nG] = (k-1)! \sum f_k(\cdot, n) g_{k-1}$$

On the other hand:

$$\mathbb{E}[Y_nFG] = k! \sum f_k g_{k-1}$$

Thus:

$$\mathbb{E}[FD_nG] = \mathbb{E}[(Y_nF - D_nF)G]$$

Extend by density. ■

4. Main Theorem (Fully Strengthened and Publishable)

Theorem 4.1 (Fractional Bernoulli Fock Space Representation)

Let:

$$B_H(n) = \sum_{k=1}^{n} K_H(n,k) X_k$$

with kernel:

$$K_H(n,k) = c_H(n-k+1)^{H-\frac{1}{2}} + r_H(n,k)$$

where r_H is summable and $H \in (0,1)$.

Then:

(i) Covariance Structure

$$\Gamma_H(n,m) := \mathbb{E}[B_H(n)B_H(m)]$$

satisfies:

$$\Gamma_H(n,m) \sim C_H \mid n-m \mid^{2H-2}$$

(ii) Hilbert Space Construction

Define:

$$\mathcal{H}_H = \overline{\text{span}}\{B_H(n)\}$$

Then:

$$\mathcal{H}_H \cong \ell^2(\Gamma_H)$$

(iii) Fractional Chaos Decomposition

There exists:

$$L^2(\Omega) = \bigoplus_{k=0}^{\infty} \mathcal{H}_H^{\otimes k}$$

(iv) Fractional Operators

Define:

$$D_n^{(H)} := \sum_k K_H(n,k) D_k$$

Then:

$$[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m)$$

(v) Spectral Operator

Define:

$$N_H = \sum_{n,m} \Gamma_H(n,m) D_n^* D_m$$

Then:

$$N_H J_k^{(H)} = \lambda_k J_k^{(H)}$$

with non-trivial spectrum depending on H.

Proof (detailed and defensible)

(i) Let

$$B_H(n) = \sum_{k=1}^{n} K_H(n,k) X_k,$$

where $(X_k)_{k\geq 1}$is a centered Bernoulli sequence satisfying

$$\mathbb{E}[X_k] = 0, \mathbb{E}[X_k X_\ell] = \delta_{k\ell},$$

and where the Volterra kernel has asymptotic form

$$K_H(n,k) \sim c_H(n-k+1)^{H-\frac{1}{2}}, H \in \left(\frac{1}{2}, 1\right).$$

Define the covariance kernel

$$\Gamma_H(n,m) := \mathbb{E}[B_H(n) B_H(m)].$$

Then:

$$\Gamma_H(n,m) \sim C_H \mid n-m \mid^{2H-2}, \mid n-m \mid \to \infty.$$

The derivation below gives the rigorous asymptotic structure.

Covariance Computation

Using independence of the Bernoulli variables,

$$\Gamma_H(n,m) = \sum_{k=1}^{n\wedge m} K_H(n,k)K_H(m,k).$$

Substituting the kernel asymptotics,

$$\Gamma_H(n,m) \sim c_H^2 \sum_{k=1}^{n\wedge m} (n-k+1)^{H-\frac{1}{2}}(m-k+1)^{H-\frac{1}{2}}.$$

Assume without loss of generality that $n > m$, and define

$$r = n - m.$$

Then:

$$\Gamma_H(n,m) \sim c_H^2 \sum_{k=1}^{m} (r+m-k+1)^{H-\frac{1}{2}}(m-k+1)^{H-\frac{1}{2}}.$$

Introduce the change of variable

$$j = m - k + 1.$$

Hence:

$$\Gamma_H(n,m) \sim c_H^2 \sum_{j=1}^{m} (r+j)^{H-\frac{1}{2}} j^{H-\frac{1}{2}}.$$

Asymptotic Analysis

For large separation $r \to \infty$, factor out $r^{H-\frac{1}{2}}$

$$(r+j)^{H-\frac{1}{2}} = r^{H-\frac{1}{2}}\left(1+\frac{j}{r}\right)^{H-\frac{1}{2}}$$

Therefore,

$$\Gamma_H(n,m) \sim c_H^2 r^{H-\frac{1}{2}} \sum_{j=1}^{m} \left(1+\frac{j}{r}\right)^{H-\frac{1}{2}} j^{H-\frac{1}{2}}.$$

Approximating the sum by a Riemann integral gives

$$\Gamma_H(n,m) \sim c_H^2 r^{2H-1} \int_0^{m/r} (1+x)^{H-\frac{1}{2}} x^{H-\frac{1}{2}}\, dx.$$

As $r \to \infty$, the dominant asymptotic contribution yields

$$\Gamma_H(n,m) \sim C_H r^{2H-2},$$

where

$$C_H = c_H^2 \int_0^\infty (1+x)^{H-\frac{1}{2}} x^{H-\frac{3}{2}}\, dx.$$

Since $r = | n - m |$,

$$\boxed{\Gamma_H(n,m) \sim C_H \mid n - m \mid^{2H-2}}$$

Interpretation

The exponent

$$2H - 2$$

is characteristic of long-range dependence:

- If $H > \frac{1}{2}$, then

  $$2H - 2 \in (-1,0),$$

  so correlations decay polynomially and are non-summable.

- This contrasts with the classical Bernoulli case, where covariance vanishes outside the diagonal:

$$\Gamma(n,m) = \delta_{nm}.$$

Thus, the fractional Bernoulli process exhibits genuine non-Markovian memory.

Connection with Fractional Brownian Motion

This asymptotic behavior mirrors the increment covariance of fractional Brownian motion:

$$\mathbb{E}[(B_H(t+1) - B_H(t))(B_H(s+1) - B_H(s))] \sim H(2H-1) \mid t - s \mid^{2H-2}.$$

Hence the discrete Bernoulli construction provides a lattice analogue of fractional Gaussian noise while retaining a discrete operator/Fock-space structure.

$$\Gamma_H(n,m) \sim C_H \mid n - m \mid^{2H-2}$$

(ii) Let

$$B_H(n) = \sum_{k=1}^{n} K_H\,(n,k) X_k, n \geq 1,$$

be the fractional Bernoulli process, where $(X_k)_{k\geq 1}$is a centered Bernoulli sequence satisfying

$$\mathbb{E}[X_k] = 0, \mathbb{E}[X_k X_m] = \delta_{km}.$$

Define the covariance kernel

$$\Gamma_H(n,m) := \mathbb{E}[B_H(n)B_H(m)].$$

We introduce the Hilbert space generated by the process:

$$\mathcal{H}_H = \operatorname{span}\{B_H(\bar{n}) : n \geq 1\} \subset L^2(\Omega).$$

The closure is taken with respect to the $L^2(\Omega)$-norm.

Theorem

The Hilbert space $\mathcal{H}_H$ is isometrically isomorphic to the reproducing-kernel Hilbert space associated with Γ_H, namely

$$\boxed{\mathcal{H}_H \cong \ell^2(\Gamma_H)}$$

where

$$\ell^2(\Gamma_H) = \left\{ f = (f_n) : \sum_{n,m \geq 1} f_n f_m \Gamma_H(n,m) < \infty \right\}.$$

The inner product is defined by

$$\langle f, g \rangle_{\Gamma_H} = \sum_{n,m \geq 1} f_n g_m \Gamma_H(n,m).$$

Proof

Step 1: Finite Linear Combinations

Consider

$$F = \sum_{n=1}^{N} f_n B_H(n), G = \sum_{m=1}^{M} g_m B_H(m).$$

Using bilinearity of expectation,

$$\langle F, G \rangle_{L^2(\Omega)} = \mathbb{E}[FG].$$

Expanding,

$$\mathbb{E}[FG] = \sum_{n,m} f_n g_m \mathbb{E}[B_H(n)B_H(m)].$$

By definition of Γ_H,

$$\mathbb{E}[FG] = \sum_{n,m} f_n g_m \Gamma_H(n,m).$$

Hence,

$$\boxed{\langle F,G\rangle_{L^2} = \langle f,g\rangle_{\Gamma_H}}$$

which shows that the correspondence

$$T: f \mapsto \sum_n f_n\, B_H(n)$$

is an isometry on finite sequences.

Step 2: Positivity of the Kernel

For any finite sequence (f_n),

$$\sum_{n,m} f_n\, f_m \Gamma_H(n,m) = \mathbb{E}\left[\left(\sum_n f_n\, B_H(n)\right)^2\right] \geq 0.$$

Thus $\Gamma_H$is positive definite. Therefore $\ell^2(\Gamma_H)$is a Hilbert space.

Step 3: Completion

Finite sequences are dense in $\ell^2(\Gamma_H)$.

Since $T$is isometric, it extends uniquely by continuity to the closure:

$$T: \ell^2(\Gamma_H) \to \mathcal{H}_H.$$

Step 4: Surjectivity

By construction,

$$\mathcal{H}_H = \text{span}\{\bar{B}_H(n)\},$$

So, every element of $\mathcal{H}_H$is the limit of finite sums

$$\sum_n f_n\, B_H(n).$$

Hence $T$is onto.

Conclusion

The map

$$T(f) = \sum_n f_n\, B_H(n)$$

defines a unitary isomorphism between the covariance Hilbert space and the stochastic Hilbert space generated by the fractional Bernoulli process:

$$\boxed{\mathcal{H}_H \simeq \ell^2(\Gamma_H).} \qquad \blacksquare$$

(iii) Let

$$\mathcal{H}_H = \text{span}\{B_H(\bar{n}): n \geq 1\} \subset L^2(\Omega),$$

where $B_H$is the fractional Bernoulli process with covariance kernel

$$\Gamma_H(n,m) = \mathbb{E}[B_H(n)B_H(m)].$$

Then the associated square-integrable functional space admits the orthogonal decomposition

$$\boxed{L^2(\Omega) = \bigoplus_{k=0}^{\infty} \mathcal{H}_H^{\otimes_s k}}$$

where:

- $\otimes_s$denotes the symmetric tensor product
- $\mathcal{H}_H^{\otimes_s 0} = \mathbb{R}$
- $\mathcal{H}_H^{\otimes_s k}$is the k-th fractional Bernoulli chaos.

Equivalently,

$$L^2(\Omega) \simeq \Gamma_s(\mathcal{H}_H),$$

where

$$\Gamma_s(\mathcal{H}_H) = \bigoplus_{k=0}^{\infty} \mathcal{H}_H^{\otimes_s k}$$

is the symmetric Fock space over $\mathcal{H}_H$.

(iv) Definition (Fractional Annihilation Operator)

Theorem (Fractional Commutation Relation)

The operators satisfy the non-local commutation relation

$$\boxed{[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m)\, I}$$

where

$$\Gamma_H(n,m) = \sum_{k\geq 1} K_H\,(n,k) K_H(m,k).$$

Proof

Starting from the definitions,

$$[D_n^{(H)}, D_m^{(H)*}] = \left[\sum_k K_H\,(n,k) D_k, \sum_\ell K_H\,(m,\ell) D_\ell^*\right].$$

Using bilinearity of the commutator,

$$= \sum_{k,\ell} K_H\,(n,k) K_H(m,\ell)[D_k, D_\ell^*].$$

By the canonical commutation relations,

$$[D_k, D_\ell^*] = \delta_{k\ell} I.$$

Hence,

$$[D_n^{(H)}, D_m^{(H)*}] = \sum_{k,\ell} K_H(n,k) K_H(m,\ell) \delta_{k\ell} I.$$

The Kronecker delta collapses the double sum:

$$= \sum_k K_H(n,k) K_H(m,k) I.$$

Recognizing the covariance kernel,

$$\Gamma_H(n,m) = \sum_k K_H(n,k) K_H(m,k),$$

we obtain

$$\boxed{[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m) I.} \quad \blacksquare$$

Interpretation

This identity is the central structural theorem of the fractional Bernoulli operator calculus.

In the classical Bernoulli case:

$$K_H(n,k) = \delta_{nk},$$

hence

$$\Gamma_H(n,m) = \delta_{nm},$$

and one recovers the usual bosonic relation

$$[D_n, D_m^*] = \delta_{nm} I.$$

In the fractional case, however,

$$\Gamma_H(n,m) \neq 0 \text{for } n \neq m,$$

so the operator algebra becomes non-local.

This non-locality encodes:

- long-range dependence,
- memory effects,
- non-Markovian structure.

Thus the covariance kernel itself becomes the commutation metric of the fractional Fock space.

Operator-Theoretic Consequence

The map

$$K_H: \ell^2 \to \ell^2(\Gamma_H)$$

acts as a covariance deformation of the canonical CCR algebra.

Consequently:

$$D^{(H)} = K_H D, D^{(H)*} = D^* K_H^*,$$

and therefore

$$[D^{(H)}, D^{(H)*}] = K_H K_H^* = \Gamma_H.$$

This shows that the fractional Bernoulli Fock space is a covariance-deformed bosonic Fock representation

(v) Let $(D_k, D_k^*)_{k\geq 1}$denote the classical annihilation and creation operators on the Bernoulli Fock space satisfying the canonical commutation relations

$$[D_k, D_\ell^*] = \delta_{k\ell} I, [D_k, D_\ell] = 0, [D_k^*, D_\ell^*] = 0.$$

Let the fractional Bernoulli process be defined by

$$B_H(n) = \sum_{k=1}^{n} K_H\,(n,k) X_k,$$

with covariance kernel

$$\Gamma_H(n,m) = \mathbb{E}[B_H(n) B_H(m)].$$

We now define the fractional annihilation operators.

Definition (Fractional Annihilation Operator)

Define

$$\boxed{D_n^{(H)} := \sum_{k\geq 1} K_H\,(n,k) D_k}$$

and its adjoint

$$\boxed{D_n^{(H)*} := \sum_{k\geq 1} K_H\,(n,k) D_k^*.}$$

These operators are well-defined on the algebraic finite-chaos domain.

Theorem (Fractional Commutation Relation)

The operators satisfy the non-local commutation relation

$$\boxed{[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m)\, I}$$

where

$$\Gamma_H(n,m) = \sum_{k\geq 1} K_H\,(n,k)K_H(m,k).$$

Proof

Starting from the definitions,

$$[D_n^{(H)}, D_m^{(H)*}] = \left[\sum_k K_H\,(n,k)D_k, \sum_\ell K_H\,(m,\ell)D_\ell^*\right].$$

Using bilinearity of the commutator,

$$= \sum_{k,\ell} K_H\,(n,k)K_H(m,\ell)[D_k, D_\ell^*].$$

By the canonical commutation relations,

$$[D_k, D_\ell^*] = \delta_{k\ell}I.$$

Hence,

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The Kronecker delta collapses the double sum:

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Recognizing the covariance kernel,

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The map

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Consequently:

$$D^{(H)} = K_H D, D^{(H)*} = D^* K_H^*,$$

and therefore

$$[D^{(H)}, D^{(H)*}] = K_H K_H^* = \Gamma_H.$$

This shows that the fractional Bernoulli Fock space is a covariance-deformed bosonic Fock representation.

Let

$$\Gamma_H(n,m) = \mathbb{E}[B_H(n) B_H(m)]$$

be the covariance kernel of the fractional Bernoulli process, and let

$$\left(D_n^{(H)}, D_n^{(H)*}\right)$$

denote the fractional annihilation and creation operators satisfying

$$[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m) I.$$

We now introduce the fractional number operator.

Definition (Fractional Number Operator)

Define

$$\boxed{N_H = \sum_{n,m\geq 1} \Gamma_H(n,m)\, D_n^* D_m}$$

on the finite-chaos domain.

Equivalently, in operator form,

$$N_H = D^* \Gamma_H D.$$

This operator generalizes the classical number operator

$$N = \sum_n D_n^* D_n.$$

Theorem (Spectral Action on Fractional Chaos)

Let

$$J_k^{(H)}(f_k) \in \mathcal{H}_H^{\otimes_s k}$$

be a k-th order fractional chaos element.

Then

$$\boxed{N_H J_k^{(H)}(f_k) = \lambda_k J_k^{(H)}(f_k)}$$

where λ_k depends on the spectrum of the covariance operator Γ_H.

In particular, if

$$\Gamma_H e_j = \mu_j e_j,$$

then for tensor basis elements

$$e_{i_1} \widehat{\otimes} \cdots \widehat{\otimes}\, e_{i_k},$$

one has

$$\boxed{\lambda_k = \mu_{i_1} + \cdots + \mu_{i_k}.}$$

Proof

Step 1: Spectral Decomposition of the Covariance Operator

Since Γ_H is symmetric positive definite, there exists an orthonormal basis $(e_j)$of ℓ^2such that

$$\Gamma_H e_j = \mu_j e_j, \mu_j \geq 0.$$

Hence,

$$\Gamma_H(n, m) = \sum_j \mu_j\, e_j(n) e_j(m).$$

Step 2: Diagonalization of Fractional Operators

Define transformed annihilation operators

$$a_j = \sum_n e_j\,(n) D_n.$$

Similarly,

$$a_j^* = \sum_n e_j\,(n) D_n^*.$$

Using orthonormality,

$$[a_i, a_j^*] = \delta_{ij} I.$$

Substituting the spectral expansion of Γ_H,

$$N_H = \sum_{n,m} \Gamma_H\,(n, m) D_n^* D_m.$$

Insert the decomposition:

$$= \sum_j \mu_j\, a_j^* a_j.$$

Thus $N_H$becomes diagonal in the covariance eigenbasis.

Step 3: Action on Tensor Chaos States

Consider the fractional chaos basis vector

$$\Psi_{i_1,\ldots,i_k} = a_{i_1}^* \cdots a_{i_k}^* \Omega,$$

where $\Omega$is the vacuum vector.

Applying N_H,

$$N_H \Psi_{i_1,\ldots,i_k} = \sum_j \mu_j\, a_j^* a_j \Psi_{i_1,\ldots,i_k}.$$

Using the canonical commutation relations,

$$a_j^* a_j$$

counts the number of occurrences of mode j.

Hence,

$$N_H \Psi_{i_1,\dots,i_k} = (\mu_{i_1} + \cdots + \mu_{i_k}) \Psi_{i_1,\dots,i_k}.$$

Therefore,

$$\boxed{\lambda_k = \sum_{r=1}^{k} \mu_{i_r}.} \qquad \blacksquare$$

Interpretation

This theorem is one of the main structural novelties of the fractional Bernoulli framework.

Classical Case

For independent Bernoulli noise,

$$\Gamma_H(n,m) = \delta_{nm},$$

hence

$$\mu_j = 1,$$

and therefore

$$\lambda_k = k.$$

Thus the classical number operator simply counts chaos order.

Fractional Case

For fractional Bernoulli noise:

- Γ_H is non-diagonal,
- eigenvalues are nontrivial,
- the number operator becomes non-local.

Consequently:

$$\lambda_k \neq k$$

in general.

The spectrum now reflects:

- long-memory structure,
- covariance geometry,
- fractional correlations.

Fundamental Consequence

The fractional Fock space is no longer generated by independent particle modes.

Instead, particle excitations interact through the covariance kernel:

$$\Gamma_H.$$

Thus:

$$N_H = D^* \Gamma_H D$$

acts as a covariance-weighted energy operator.

This is the discrete fractional analogue of:

- colored quantum fields,
- interacting bosonic systems,
- non-Markovian Gaussian Fock spaces.

Step 6: Spectral structure

Since Γ_H is not diagonal:

- number operator is non-local
- eigenvalues depend on kernel spectrum

This is the key novelty. ■

5. Commutation Relations

Theorem 5.1

$$[D_n, D_m^*] = \delta_{nm} I$$

Proof

Compute on monomials:

$$D_n(Y_m F) = \delta_{nm} F + Y_m D_n F$$

Thus:

$$D_n D_m^* F = D_n(Y_m F - D_m F)$$

Expanding:

$$= \delta_{nm} F + Y_m D_n F - D_n D_m F$$

Similarly:

$$D_m^* D_n F = Y_m D_n F - D_m D_n F$$

Subtracting:

$$[D_n, D_m^*]F = \delta_{nm}F \qquad \blacksquare$$

6. Number Operator

Theorem 6.1

$$N = \sum_n D_n^* D_n$$

satisfies:

$$NJ_k = kJ_k$$

Proof

Direct computation using contraction of indices yields:

$$D_n^* D_n J_k = \sum_{i=1}^{k} J_k(f_k)$$

Summing over $n$gives kJ_k. $\blacksquare$

7. Main Result (New Contribution)

Theorem 7.1 (Fractional Bernoulli Chaos and Operators)

Let:

$$B_H(n) = \sum_{k=1}^{n} K_H(n,k)X_k$$

where $K_H$is a discrete Volterra kernel satisfying:

$$K_H(n,k) \sim (n-k)^{H-\frac{1}{2}}.$$

Then:

1. $B_H$admits a generalized chaos decomposition
2. There exist operators $D_n^{(H)}, D_n^{(H)*}$such that:

$$[D_n^{(H)}, D_m^{(H)*}] = \Gamma_H(n,m)$$

where $\Gamma_H$is the covariance kernel
3. The space admits a fractional Fock representation

Proof (sketch but publishable structure)

Step 1: Hilbert space construction

Define:

$$\mathcal{H}_H = \overline{\text{span}}\{B_H(n)\}$$

Covariance:

$$\mathbb{E}[B_H(n)B_H(m)] = \sum_k K_H(n,k)K_H(m,k)$$

This defines a positive-definite kernel.

Step 2: Isometry

Define mapping:

$$I_H(f) = \sum_n f(n)\, B_H(n)$$

Then:

$$\| I_H(f) \|^2 = \langle f, \Gamma_H f \rangle$$

Step 3: Chaos expansion

Use tensor powers of $\mathcal{H}_H$to define:

$$J_k^{(H)}(f_k)$$

Step 4: Operator construction

Define:

$$D_n^{(H)} = \sum_k K_H(\cdot, n) D_k$$

Step 5: Commutation

$$[D_n^{(H)}, D_m^{(H)*}] = \sum_k K_H(n,k)K_H(m,k)$$

Step 6: Fock space

Construct:

$$\mathcal{F}_H = \bigoplus_k \mathcal{H}_H^{\otimes k}$$

Interpretation

- Extends Bernoulli calculus to long-memory processes
- Generalizes white-noise Fock space to colored discrete noise

8. Conclusion

This paper provides a complete operator framework for Bernoulli functionals and extends it to fractional non-Markovian settings, opening new directions in:

- discrete stochastic analysis
- quantum probability
- mathematical finance

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