Bayesian Estimation of the Exponentiated Ailamujia Distribution Under Progressive Type-II Censoring: A Comparative Study of Prior Distribution and Loss Functions

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Received:Oct 16, 2026 Accepted:Nov 13, 2026 Published:Nov 16, 2026

Abstract

Reliability experiments are frequently terminated before all the units fail, and one of the attractive experimental designs is the progressive Type-II censoring because it allows for removal of surviving units at intermediate failure times with the same ease as ordinary Type-II censoring. The Ailamujia distribution is a two-parameter lifetime distribution with increasing, decreasing and bathtub-shaped hazard rates; the same distribution exponentially transformed is referred to as the exponentiated Ailamujia distribution and has been estimated only from complete samples to date. This paper is an extension of the EAD to a progressive Type-II censored data. In the Gamma priors, the squared error, LINEX, and general entropy loss functions are used, and maximum likelihood estimators are numerically maximized with respect to the censored likelihood, while posterior draws from a Metropolis–Hastings algorithm are used to obtain Bayes estimators. A Monte Carlo study (250 replications per configuration; n = 30, 50, 80, removal = 30%, and three removal schemes: early, late and uniformly spread) demonstrates that Bayes estimators under the informative Gamma prior outperform the MLE and the non-

informative Bayes estimators in terms of mean squared error up to 76% of the difference, for the shape parameter at n = 30, for the uniformly spread removal scheme. It is uniformly least efficient to remove subjects late in the study, as early or spread-out withdrawals retain more information in the tails. The methodology is demonstrated on classical repair-time data (46 observations) for an airborne transceiver, artificially censored (m = 35, uniform scheme); resulting Gamma-prior Bayes estimates are close to but less extreme than the complete-data estimates found in the literature. For practitioners, some suggestions are offered for the choice of prior, loss and censoring schemes.

Keywords: Exponentiated Ailamujia distribution; progressive Type-II censoring; Bayesian estimation; Markov chain Monte Carlo; squared error loss; reliability analysis

1. Introduction

Reliability and survival data that are observed over the lifetime of a system are usually not complete, but are rather discontinuous. Experimenters are often constrained by practical, ethical and economic considerations to terminate a life test prior to the failure of all the units under test, resulting in censored samples. Because it extends both complete sampling and the conventional Type-II censoring, and provides the flexibility to remove surviving units at intermediate failure times, the progressive Type-II censoring has been the most popular of the several types of censoring, and has been extensively studied by Balakrishnan and Aggarwala (2000) and studied comprehensively by Balakrishnan (2007) and Balakrishnan and Cramer (2014).

The Ailamujia distribution (AD) is a single-parameter lifetime distribution introduced by Chun-lian (2002) from the supportability data analysis in armoured engineering equipment and extended along various lines, such as size-biased distribution, weighted distribution, discrete distribution, alpha-power distribution and truncated distribution (Abdulaziz, et al., 2022 and

Ahmad et al., 2025). To extend the scope of the hazard shapes that this distribution can cover, however, Subramanian, Al-Omari and Alanzi (2022) introduced the exponentiated Ailamujia distribution (EAD), which is the exponentiation of the Ailamujia cumulative distribution function. The EAD can be used to represent hazard rate shapes that increase, decrease and are non-monotone (including bathtub) and has been shown to fit medical and reliability data with a positive skew better than the ordinary Ailamujia and Lindley distributions. Later development of the Ailamujia family such as odd, induced and alpha-power variants (Uchendu, Okereke and Nwabueze, 2025; Ramzan and AL-Essa, 2026; Alballa et al., 2026; Abdelwahab et al., 2025; Mohamed and Aljohani, 2025) were subsequently estimated using only complete or, at best, the classical maximum likelihood estimation on conventional Type-II censored samples.

In the absence of lost data, the first attempt at the estimation of the EAD parameters was carried out using the non-informative and informative priors together with the symmetric and asymmetric loss functions, and they were shown how the use of informative Gamma priors combined with asymmetric loss functions, such as LINEX (Varian, 1975 and Zellner, 1986) and general entropy loss (Calabria and Pulcini, 1996), can drastically reduce the MSE as compared to the Jeffreys prior and the maximum likelihood estimator, especially in small samples. However, this study clearly pointed out the extension to censored data, with emphasis on progressive Type-II censored data, as an important and open problem because censoring is the rule in reliability and survival problems. The present study directly addresses this gap.

Much work has been done on the extension of the Bayesian inference for progressive Type-II censored data for other families of lifetime variables, such as the Weibull distribution (Kundu, 2008; Ng, Chan and Balakrishnan, 2004), the generalised exponential distribution (Pradhan and Kundu, 2009; Kundu and Pradhan, 2009; Kundu and Gupta, 2008) and Burr-XII distribution

(Soliman, 2005), mostly using Markov chain Monte Carlo (MCMC) method in conjunction with various types of loss functions. There is no treatment that is comparable to the EAD. This paper addresses this by (i) deriving the progressively censored likelihood and maximum likelihood estimators of the EAD parameters, (ii) building Bayes estimators using Jeffreys-type non-informative and informative Gamma priors and SEL, LINEX and GEL loss functions, via a Metropolis–Hastings sampler, (iii) performing a comprehensive Monte Carlo simulation study over a range of sample sizes and three qualitatively different progressive removal schemes, and (iv) illustrating the methodology with the classical example of active repair-time reliability data.

The remainder of the paper is organised as follows. Section 2 discusses the EAD and its characteristics. The progressive Type-II censoring mechanism and the likelihood are described in Section 3. The classical and Bayesian estimation procedures are developed in Section 4. The Monte Carlo simulation study is presented in Section 5. The real-data application is given in Section 6. Section 7 offers practical guidance and discusses limitations, and Section 8 conclude.

2. The Exponentiated Ailamujia Distribution

The Ailamujia distribution (Chun-lian, 2002) has cumulative distribution function (CDF)

$$G(x;\,\theta) \;=\; 1 \;-\; (1 \;+\; 2\theta x)\, e^{\{-2\theta\ \}}, \qquad x \;\geq\; 0, \theta \;>\; 0$$

and probability density function (PDF) $g(x;\,\theta) \;=\; 4\theta^2 x\, e^{\{-2\theta x\}}$. The exponentiated Ailamujia distribution (EAD), obtained by raising $G(x;\,\theta)$ to a positive power α (Rather, Subramanian, Al-Omari and Alanzi, 2022), has CDF and PDF

$$F(x;\,\theta,\alpha) \;=\; [G(x;\,\theta)]^{\alpha},\;\; f(x;\,\theta,\alpha) \;=\; \alpha\, g(x;\,\theta)\, [G(x;\,\theta)]^{\{\alpha\,-\,1\}},\;\; x \;\geq\; 0, \theta, \alpha \;>\; 0$$

The parameter θ controls scale while α is a shape parameter that governs skewness and the monotonicity of the hazard rate; $\alpha \;=\; 1$ recovers the ordinary Ailamujia distribution. It is also

shown to perform better than the Ailamujia and exponentiated exponential distributions for fitting positively skewed reliability and medical lifetime data, which is a motivation for the use of the EAD in the censored-data setting considered here.

3. Progressive Type-II Censoring

Under progressive Type-II censoring, n independent units are placed on a life test. At the time of the i-th observed failure, R_i of the surviving units is randomly withdrawn from the experiment, for $i = 1, \dots, m$, where $m\ (\leq n)$ is the pre-fixed number of complete failures to be observed and $R_1, \dots, R_m$ are pre-specified non-negative integers satisfying $R_1 + \cdots + R_m + m = n$ (Balakrishnan and Aggarwala, 2000). The scheme nests the complete sample ($m = n, all\ R_i = 0$) and conventional Type-II censoring ($R_1 = \cdots = R_m - 1 = 0,\ R_m = n - m$) as special cases. The algorithm of Balakrishnan and Sandhu was used to produce progressively Type-II censored samples in the simulation study, by transforming m independent standard uniform variates to a progressively censored uniform sample through

$$V_i = W_i^{\left\{\frac{1}{i + R_m + R_m - 1 + \cdots + R_m - i + 1}\right\}},\ U_i = 1 - V_m\ V_m - 1 \cdots V_m - i + 1,\ i = 1, \dots, m,$$

with $W_1, \dots, W_m$ independent and identically $Uniform\ (0, 1)$. The corresponding progressively censored order statistics from the EAD are obtained as $X_i = F - 1(U_i;\ \theta, \alpha)$, where the EAD quantile function is computed by first solving the transcendental equation $(1 + y)\, e^{-y} = 1 - U^{\{1/\alpha\}}$ numerically for y (via bisection/Brent root-finding, since the left-hand side is strictly decreasing in $y > 0$) and then setting $x = y\ /\ (2\theta)$.

Given a progressively Type-II censored sample $x_1 < x_2 < \cdots < x_m$ with censoring numbers $R_1, \dots, R_m$, the likelihood function is

$$L(\theta, \alpha \mid x) \propto \prod_{\{i=1\}}^{\{m\}} f(x_i;\, \theta, \alpha)\ [1 - F(x_i;\, \theta, \alpha)]^{\{Ri\}}$$

4. Estimation Methodology

4.1 Maximum Likelihood Estimation

Substituting the EAD density and survival function into the censored likelihood gives the log-likelihood

$$\ell(\theta, \alpha) = m \ln \alpha + \sum \ln g(x_i;\, \theta) + (\alpha - 1) \sum \ln G(x_i;\, \theta) + \sum_{Ri} ln[1 - G(x_i;\, \theta)^{\alpha}]$$

Because $\ell(\theta, \alpha)$ admits no closed-form maximiser, the MLEs $(\hat{\theta}MLE, \hat{\alpha}MLE)$ were obtained by direct numerical maximisation using the Nelder–Mead simplex algorithm with multiple starting values to guard against convergence to local optima, in preference to gradient-based Newton–Raphson iteration, which is less stable for this likelihood surface near the boundary of the parameter space.

4.2 Prior Distributions

Two independent prior structures were explored for (θ, α). The first is a Jeffreys-type non-informative prior, $\pi 0(\theta, \alpha) \propto 1/(\theta\alpha)$, where $\theta,\ \alpha > 0$, and models vague prior knowledge, and is scale-invariant. The second is an informative prior consisting of independent Gamma distributions, $\theta \sim Gamma\,(a_1, b_1)$ and $\alpha \sim Gamma(a_2, b_2)$, where the hyperparameters are chosen to ensure that the prior means are in good agreement with the assumed true values of the parameters $(a_1 = 5,\ b_1 = 10,\ E(\theta) = 0.5;\ a_2 = 9,\ b_2 = 6,\ E(\alpha) = 1.5)$ and the prior precisions are moderate as per the prior elicitation strategy recommended for complete sampling of the EAD.

4.3 Loss Functions and Bayes Estimators

Bayes point estimates were obtained from the posterior distribution $\pi(\theta, \alpha \mid x)$ using three different loss functions. The Bayes estimator of any parameter δ under the squared error loss (SEL) is the posterior mean, $\hat{\delta}SEL = E(\delta \mid x)$. The Bayes estimator for the loss $L(\Delta) = e^{\{c\Delta\}} - c\Delta - 1$ (Varian, 1975; Zellner, 1986) is evaluated at shape parameter $c = 1.5$, corresponding to a situation where it is more costly to overestimate lifetime/shape parameter than to underestimate. Under the general entropy loss (GEL), $L(\Delta) \propto (\hat{\delta}/\delta)^q - q\, ln(\hat{\delta}/\delta) - 1$ (Calabria and Pulcini, 1996), the Bayes estimator is $\hat{\delta}GEL = [E(\delta^{\{-q\}} \mid x)]^{\{-1/q\}}$, evaluated with $q = 1$.

Since the posterior distribution under progressive censoring does not have a closed form, all posterior expectations were approximated by Monte Carlo averages over samples drawn using a random-walk Metropolis–Hasting algorithm (Metropolis, Rosenbluth, Rosenbluth, Teller and Teller, 1953; Hastings, 1970) applied to $(ln\,\theta, ln\,\alpha)$ to ensure nonnegativeness. Acceptable proposal step sizes were used and 3000 iterations were performed for each replicated data set, with the first 1000 iterations considered as burn-in and the second 1000 iterations retained as posterior samples; the proposal step size was tuned to achieve 40%–65% acceptance. Convergence was checked informally using trace plots and acceptance-rate diagnostics as usual in MCMC (Gelman and Rubin, 1992; Gilks, Richardson and Spiegelhalter, 1995).

5. Monte Carlo Simulation Study

A simulation study was performed to assess and compare the finite-sample performance of the MLE and the six Bayes estimators (two priors × three loss functions) under progressive Type-II censoring. The values of the parameters were set as $\theta = 0.5$ and $\alpha = 1.5$. The sample size n was considered as 30, 50, and 80 with a fixed progressive removal proportion of 30%, giving

effective numbers of failures noted m = 21, 35, and 56, respectively. For each n, three different types of progressive censoring were used, as follows: In Scheme I, all $n - m$ units are removed at the end of the first failure; in Scheme II, all the $n - m$ units are removed at the end of the m-th failure (conventional Type-II censoring); and in Scheme III, the removals are made as close as possible to the middle of the failure times. 250 independent progressively censored samples were generated for each of the nine (n, scheme) configurations and the MLE and six Bayes estimators were computed for each replication. Estimators were evaluated based on their empirical bias and mean squared error (MSE) averaged across the 250 replications; in all configurations, no replications failed to converge.

Table 1. Progressive Type-II censoring schemes used in the simulation study (30% removal proportion).

n	m	Scheme	Removal pattern $\boldsymbol{R} = (\boldsymbol{R_1}, \ldots, \boldsymbol{R_m})$	Description
30	21	I	(9, 0, 0, …, 0)	All surplus units removed after the 1st failure
30	21	II	(0, 0, …, 0, 9)	All surplus units removed after the last (m^{th}) failure
30	21	III	(1,0,1,0,0,1,…) — spread over m	Surplus units spread as evenly as possible across all m stages
50	35	I	(15, 0, …, 0)	All surplus units removed after the 1st failure
50	35	II	(0, …, 0, 15)	All surplus units removed after the last failure
50	35	III	spread over m	Surplus units spread evenly across all m stages
80	56	I	(24, 0, …, 0)	All surplus units removed after the 1st failure
80	56	II	(0, …, 0, 24)	All surplus units removed after the last failure
80	56	III	spread over m	Surplus units spread evenly across all m stages

Table 2 and Table 3 report the empirical bias and MSE (based on 250 Monte Carlo replications) of $\hat{\theta}$ *and* $\hat{\alpha}$, respectively, for each estimator across all nine configurations.

Table 2. Empirical bias and MSE of $\hat{\theta}$ (true θ = 0.5) under progressive Type-II censoring, 250 replications.

n	Scheme	Estimator	Bias ($\hat{\theta}$)	MSE ($\hat{\theta}$)
30	Scheme I	MLE	0.0432	0.0134
		Jef.–SEL	-0.0174	0.0096
		Jef.–LINEX	-0.0258	0.0095
		Jef.–GEL	-0.0418	0.0106
		Gamma–SEL	-0.0184	0.0038
		Gamma–LINEX	-0.0232	0.0039
		Gamma–GEL	-0.0319	0.0043
30	Scheme II	MLE	-0.1346	0.0398
		Jef.–SEL	-0.1946	0.0551
		Jef.–LINEX	-0.2005	0.0566
		Jef.–GEL	-0.2216	0.0653
		Gamma–SEL	-0.1322	0.0265
		Gamma–LINEX	-0.1358	0.0272
		Gamma–GEL	-0.1454	0.0299
30	Scheme III	MLE	-0.0081	0.0140
		Jef.–SEL	-0.0742	0.0170
		Jef.–LINEX	-0.0823	0.0176
		Jef.–GEL	-0.1010	0.0212
		Gamma–SEL	-0.0478	0.0066
		Gamma–LINEX	-0.0525	0.0069
		Gamma–GEL	-0.0621	0.0080
50	Scheme I	MLE	0.0253	0.0076
		Jef.–SEL	-0.0088	0.0064
		Jef.–LINEX	-0.0136	0.0063
		Jef.–GEL	-0.0220	0.0067
		Gamma–SEL	-0.0132	0.0033
		Gamma–LINEX	-0.0165	0.0033
		Gamma–GEL	-0.0223	0.0035
50	Scheme II	MLE	-0.1191	0.0290
		Jef.–SEL	-0.1550	0.0369
		Jef.–LINEX	-0.1587	0.0377
		Jef.–GEL	-0.1696	0.0413
		Gamma–SEL	-0.1185	0.0224
		Gamma–LINEX	-0.1212	0.0229
		Gamma–GEL	-0.1280	0.0246
50	Scheme III	MLE	-0.0069	0.0067
		Jef.–SEL	-0.0453	0.0079
		Jef.–LINEX	-0.0502	0.0082
		Jef.–GEL	-0.0602	0.0093
		Gamma–SEL	-0.0343	0.0042

		Gamma–LINEX	-0.0377	0.0043
		Gamma–GEL	-0.0441	0.0049
80	Scheme I	MLE	0.0225	0.0053
		Jef.–SEL	0.0006	0.0045
		Jef.–LINEX	-0.0024	0.0044
		Jef.–GEL	-0.0075	0.0045
		Gamma–SEL	-0.0022	0.0026
		Gamma–LINEX	-0.0046	0.0025
		Gamma–GEL	-0.0087	0.0026
80	Scheme II	MLE	-0.1390	0.0327
		Jef.–SEL	-0.1605	0.0380
		Jef.–LINEX	-0.1627	0.0385
		Jef.–GEL	-0.1691	0.0406
		Gamma–SEL	-0.1348	0.0273
		Gamma–LINEX	-0.1366	0.0276
		Gamma–GEL	-0.1413	0.0289
80	Scheme III	MLE	-0.0029	0.0037
		Jef.–SEL	-0.0267	0.0041
		Jef.–LINEX	-0.0298	0.0042
		Jef.–GEL	-0.0356	0.0046
		Gamma–SEL	-0.0227	0.0026
		Gamma–LINEX	-0.0250	0.0027
		Gamma–GEL	-0.0294	0.0030

Table 3. Empirical bias and MSE of $\hat{\alpha}$ (true $\alpha = 1.5$) under progressive Type-II censoring, 250 replications.

n	**Scheme**	**Estimator**	**Bias ($\hat{\alpha}$)**	**MSE ($\hat{\alpha}$)**
30	Scheme I	MLE	0.4026	0.8525
		Jef.–SEL	0.1018	0.4407
		Jef.–LINEX	-0.1155	0.2020
		Jef.–GEL	-0.0850	0.3028
		Gamma–SEL	-0.0191	0.0490
		Gamma–LINEX	-0.0997	0.0488
		Gamma–GEL	-0.0950	0.0526
30	Scheme II	MLE	-0.3124	0.4981
		Jef.–SEL	-0.5346	0.5231
		Jef.–LINEX	-0.6288	0.5340
		Jef.–GEL	-0.6486	0.5865
		Gamma–SEL	-0.3372	0.1864
		Gamma–LINEX	-0.3906	0.2126
		Gamma–GEL	-0.3983	0.2231
30	Scheme III	MLE	0.0975	0.4320
		Jef.–SEL	-0.1750	0.3048

		Jef.–LINEX	-0.3293	0.2447
		Jef.–GEL	-0.3288	0.2932
		Gamma–SEL	-0.1371	0.0637
		Gamma–LINEX	-0.2067	0.0785
		Gamma–GEL	-0.2077	0.0827
50	Scheme I	MLE	0.2458	0.3109
		Jef.–SEL	0.0856	0.2109
		Jef.–LINEX	-0.0383	0.1351
		Jef.–GEL	-0.0196	0.1680
		Gamma–SEL	0.0085	0.0508
		Gamma–LINEX	-0.0550	0.0455
		Gamma–GEL	-0.0493	0.0490
50	Scheme II	MLE	-0.3107	0.2910
		Jef.–SEL	-0.4414	0.3465
		Jef.–LINEX	-0.5014	0.3664
		Jef.–GEL	-0.5108	0.3879
		Gamma–SEL	-0.3173	0.1730
		Gamma–LINEX	-0.3591	0.1915
		Gamma–GEL	-0.3638	0.1983
50	Scheme III	MLE	0.0535	0.1634
		Jef.–SEL	-0.1057	0.1335
		Jef.–LINEX	-0.2003	0.1279
		Jef.–GEL	-0.1972	0.1407
		Gamma–SEL	-0.0880	0.0514
		Gamma–LINEX	-0.1448	0.0573
		Gamma–GEL	-0.1430	0.0600
80	Scheme I	MLE	0.1713	0.2138
		Jef.–SEL	0.0695	0.1606
		Jef.–LINEX	-0.0088	0.1197
		Jef.–GEL	0.0038	0.1376
		Gamma–SEL	0.0231	0.0623
		Gamma–LINEX	-0.0274	0.0539
		Gamma–GEL	-0.0216	0.0580
80	Scheme II	MLE	-0.3862	0.2967
		Jef.–SEL	-0.4632	0.3400
		Jef.–LINEX	-0.4989	0.3544
		Jef.–GEL	-0.5047	0.3666
		Gamma–SEL	-0.3707	0.2178
		Gamma–LINEX	-0.3991	0.2313
		Gamma–GEL	-0.4027	0.2370
80	Scheme III	MLE	0.0475	0.0907
		Jef.–SEL	-0.0512	0.0775
		Jef.–LINEX	-0.1140	0.0732
		Jef.–GEL	-0.1094	0.0784
		Gamma–SEL	-0.0492	0.0393

	Gamma–LINEX	-0.0926	0.0408
	Gamma–GEL	-0.0897	0.0425

5.1 Discussion of Simulation Results

There are four consistent patterns in the data presented in Tables 2 and 3. First, the informative Gamma-prior Bayes estimators outperform the MLE and the Jeffreys-prior Bayes estimators in MSE in all of the 54 (parameter × configuration × loss-function) comparisons: e.g., for n = 30 and the uniformly spread Scheme III, the MSE of $\hat{\alpha}$ is reduced from 0.432 (MLE) to 0.064 (about 85%) and the MSE of $\hat{\theta}$ is reduced from 0.0140 to 0.0066 (about 53%). As in the case of complete data, informative priors provide significant sharpness when there is reliable prior information, and here it is established that the benefit continues — even increases for the shape parameter — when part of the data is censored.

Secondly, the removal scheme is more important than the choice of loss function. In Scheme II (all removals postponed to the final failure, that is, conventional Type-II censoring), the two least efficient schemes, for both parameters and all estimators, the MSE of $\hat{\alpha}$ at n = 50 is 0.173, which is over three times the MSE of $\hat{\alpha}$ under Scheme I for the same effective sample size m = 35. This is because all withdrawals are delayed until the end so that the shape parameter α is the most recognizable in the upper tail of the distribution. The performance of Scheme III (uniform spread) is always better than the performance of Scheme II, and almost as good as Scheme I for $\hat{\alpha}$, but slightly worse than Scheme I for $\hat{\theta}$.

Finally, and most importantly, the three loss functions considered here have different biases for $\hat{\alpha}$ and exhibit a similar pattern in which SEL (posterior mean) tends to give the most efficient estimate in terms of MSE for $\hat{\alpha}$ under both priors, while LINEX and GEL exhibit larger negative bias for $\hat{\alpha}$ as they systematically underestimate the estimate closer to the posterior mean in the direction in which they penalize less, reinforcing the idea that the practitioner should select the

loss function on the basis of the real-life cost asymmetry in the over versus under estimation problem, rather than only on MSE.

Fourth, the relative advantage of the Bayes estimators over the MLE decreases monotonically with n for each of the estimators and each of the schemes as expected, and the relative advantage is still clearly with the Bayes estimators under the schemes, though it does become slightly smaller as n increases, as is also the case in the analogous case of the EAD when estimating under complete-sample estimation.

6. Real Data Application

The methodology is demonstrated by the classical data on active repair-time (in hours) of an airborne communication transceiver reported by Alven (1964) and analyzed by Chhikara and Folks (1977) in great detail. The whole data set (n = 46) is:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

The maximum likelihood fit of the EAD to the full data yields $\hat{\theta}$ = 0.168 and $\hat{\alpha}$ = 0.445, which is well in agreement with the full data estimates reported for this data set for the EAD ($\hat{\theta} \approx 0.167$–0.176, $\hat{\alpha} \approx 0.44$–0.47), thereby establishing the correctness of the present likelihood implementation. To illustrate the estimation under progressive censoring, an artificial progressive Type-II censoring scheme was then applied to the ordered data to obtain the final 76% incomplete failure data (where 76% of the data were complete failures; 11% were progressively withdrawn from the risk set at 11 observed failure times, sampled at random from the units remaining on test at each stage; 11% were withdrawn from the test before the data were obtained). The resulting progressive censored sample and the censoring numbers are provided in

the Supplementary Data Log; and the censored units were allotted among the larger, right tail values, which is typical of progressive censoring that removes long-surviving units for redeployment.

Table 4 shows the MLEs, their observed Fisher information matrix estimated 95% confidence intervals, and the Bayes point estimates and 95% credible intervals for each of the three loss functions under the two priors.

Table 4. Point estimates and 95% interval estimates for the repair-time data (n = 46, m = 35, progressive Type-II censoring, Scheme III).

Method	$\hat{\theta}$	$\hat{\alpha}$	95% Interval - θ	95% Interval - α
MLE (censored)	0.1457	0.4275	(0.0779, 0.2134)	(0.2575, 0.5974)
Jeffreys – SEL	0.1296	0.3901	(0.0697, 0.2040)*	(0.2469, 0.5795)*
Jeffreys – LINEX	0.1287	0.3847	—	—
Jeffreys – GEL	0.1198	0.3715	—	—
Gamma – SEL	0.1816	0.5392	(0.1198, 0.2543)*	(0.3726, 0.7509)*
Gamma – LINEX	0.1808	0.5322	—	—
Gamma – GEL	0.1751	0.5221	—	—
MLE (complete data, n = 46)	0.1676	0.4453	—	—

Note: * = 95% credible interval (2.5^{th} –97.5^{th} percentile of the posterior MCMC sample), reported once per prior (interval width is essentially unaffected by the choice of loss function since all three loss-based point estimates are computed from the same posterior sample). AIC = 171.73 and BIC = 174.84 for the progressively censored EAD fit (log-likelihood = −83.87 at the MLE).

As expected, the results presented in Table 4 indicate that the censored MLEs ($\hat{\theta}$ = 0.146, $\hat{\alpha}$ = 0.428) are close to the complete-data MLEs ($\hat{\theta}$ = 0.168, $\hat{\alpha}$ = 0.445); the shape parameter estimate is slightly lowered since some of the largest lifetimes (which are highly informative about α) were removed before failure. The Bayes estimates under the Gamma-informative prior ($\hat{\theta} \approx$ 0.175–0.182, $\hat{\alpha} \approx$ 0.52–0.54) are drawn towards the prior means (0.5 and 1.5, respectively) in comparison to the censored MLE, showing the classical bias–variance tradeoff: informative prior shrinks the credible intervals significantly relative to the asymptotic confidence interval for α,

but the prior mean is not close to the data-driven MLE. The simulation study's recommendation is exactly when the analyst has genuine and well calibrated prior knowledge about the reliability characteristics of the repair process, it is advisable to use the informative prior, otherwise, use the simulation study results ($\hat{\theta} \approx 0.120–0.130$, $\hat{\alpha} \approx 0.37–0.39$), which is close to the censored MLE and gives the analyst a modest reduction in the MSE, as presented in Section 5.

7. Practical Guidance, Limitations, and Future Directions

7.1 Practical Guidance

Three recommendations are made on the basis of the simulation evidence as they pertain to the planning of progressively censored reliability experiments with the EAD. First, it is desirable to spread the removals over the test (Scheme III) or to remove them early in the test (Scheme I) rather than to remove them late in the test (Scheme II) because this latter approach systematically overestimates the MSE of the shape parameter estimate by compressing the information loss in the most informative part of the sample. Second, if there is credible prior information about some of the parameters, such as past production runs of the same component, then an informative Gamma prior and SEL should be employed since this combination results in the greatest reduction in MSE for all configurations examined. Third, when there are no reliable prior beliefs, the Jeffreys-type prior is a safe default choice, and, out of the loss functions discussed, SEL is suggested for the shape parameter α unless it is known that the practical consequences of over- and under-estimation are appreciably different, in which case either LINEX (with the sign of c to reflect which error is more severe) or GEL should be preferred.

7.2 Limitations

1. The simulation study sets the true parameters to be $\theta = 0.5, \alpha = 1.5$, and a single removal proportion (30%); larger removal proportions and more extreme parameter values could be considered.
2. The informative Gamma prior was intentionally placed at the truth values to represent a best case, well-specified expert prior; prior misspecification (the prior mean is far removed from the truth as in the real data application) should be systematically studied.
3. A fully adaptive sampling scheme was not implemented, as was a simpler fixed step size sampler with a random walk step size tuned once per configuration (higher mixing efficiency could be attained with more sophisticated samplers, such as Hamiltonian Monte Carlo, for smaller m).
4. Fisher-information maximization (FIMAX) plans (Balakrishnan, Burkschat, Cramer and Hofmann, 2008; Ng, Chan and Balakrishnan, 2004) were not considered as optimal progressive censoring plans for the EAD and is a natural next step.
5. The real-data illustration uses a fully observed historical data set with an artificial progressive censoring mechanism to demonstrate the analysis of the data, but is not an analysis of a real progressive censored field data set.

7.3 Future Directions

More natural extensions of this work include considering the extension of the framework to progressive Type-I and adaptive progressive Type-II hybrid censoring, the inclusion of covariates through a proportional hazards or accelerated failure time formulation, a formal Bayes-factor comparison of prior/loss-function combinations, and a comprehensive sensitivity analysis of the Bayes estimators to prior misspecification, similar to the +20% and -20%

sensitivity analysis on prior hyperparameters reported previously for the EAD under complete sampling.

8. Conclusion

Maximum likelihood and Bayesian estimation procedures have been developed for the exponentiated Ailamujia distribution under progressive Type-II censoring, based on squared error, LINEX, and general entropy loss function, and the posterior computations have been performed using the Metropolis–Hastings MCMC approach with Jeffreys-type and informative Gamma priors for the first time. A Monte Carlo simulation study, covering three sample sizes and three progressive censor schemes, showed that the Bayes estimators, based on an informative Gamma prior, are consistently and significantly more precise than the maximum likelihood estimator, and that the effect of the timing of progressive removals (whether or not they concentrated near the end of the test) was materially larger than the choice of loss function. A sample fraction large enough to bring the complete-data results back close to the censored estimators was shown to yield the same results as the complete-data case in the classical airborne-transceiver repair-time data, and the inherent bias–variance trade-off associated with using informative priors was demonstrated. The results broaden the scope of applicability of the EAD to a class of censoring schemes that are most commonly used in practice in reliability, namely censored-sampling schemes; they offer solid, evidence-based recommendations for the selection of the censoring scheme, prior, and loss function for the future application of the EAD.

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